

ERROR BOUNDS FOR POLYNOMIAL AND SPLINE INTERPOLATION

By

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To my wife, Nadia

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The present dissertation is motivated by a desire to have a more precise knowledge of asymptotic approximation error than that given by best order of approximation. It owes its inspiration to a paper by G. Birkhoff and A. Priver concerning error bounds for derivatives of Hermite interpolation and a paper of C. A. Hall and W. W. Meyer concerning error bounds for cubic splines.

In Chapter One we consider well known results concerning interpolation, polynomial approximation and error analysis of spline approximation. The results given here are meant to provide a context for the theorems given in later chapters. In Chapters Two and Three we consider the problem of best error bounds for derivatives in two point Birkhoff interpolation problems.

Chapter Four presents the problems of existence, uniqueness, explicit representation, and the problem of convergence for fourth degree splines. Moreover we also consider the problem of optimal pointwise error bounds for functions  $f \in C^{(5)}[0,1]$ . In Chapter Five our main object is to sharpen the error bounds obtained earlier by Marsden concerning quadratic spline interpolation. By doing so we obtain in some special cases error bounds that are in fact optimal.

## CHAPTER ONE INTRODUCTION

The purpose of this chapter is to provide a context for the results derived in succeeding chapters. In order to show some of the important achievements in approximation by polynomials, we discuss briefly the Lagrange and Hermite-Fejér interpolations, which match a given function at any finite number of distinct points. After exploring the question of computational stability of a given interpolation, we discuss in some detail the problem of best order of approximation by polynomials as initiated by S. N. Bernstein [1912], D. Jackson [1930], and A. Zygmund [1968].

In contrast to high order approximation by a single polynomial, we next consider in great detail the problem of approximating a given function  $f(x)$  defined on  $[a,b]$  by the interpolatory piecewise polynomials known as splines. Special attention is given to the problem of approximating by piecewise cubic and piecewise parabolic splines. The study of these splines motivates us to also study two point Hermite and Birkhoff interpolations.

### Lagrange and Hermite-Fejér Interpolation

Let  $X$  denote an infinite triangular matrix with all entries in  $[-1, 1]$

$$(1.1.1) \quad X: \begin{array}{ccccccc} & & x_{0,0} & & & & \\ & & x_{0,1} & x_{1,1} & & & \\ & & \cdot & & & & \\ & & \cdot & & & & \\ & x_{0,n} & x_{1,n} & \cdot & \cdot & \cdot & x_{n,n} \end{array}$$

We denote by  $L_n[f, x; X]$  the Lagrange polynomial of interpolation of degree  $\leq n$  which coincides with  $f(x)$  in the nodes  $x_{kn}$  ( $k = 0, 1, \dots, n$ ). Then

$$(1.1.2) \quad L_n[f, x; X] = \sum_{k=0}^n f(x_{kn}) l_{kn}(x)$$

where

$$(1.1.3) \quad l_{kn}(x) = \frac{\omega_n(x)}{(x - x_{kn}) \omega_n'(x_{kn})},$$

$$\omega_n(x) = \prod_{k=0}^n (x - x_{kn}).$$

It is known from the results of G. Faber and S. N. Bernstein that no matrix  $X$  is effective for the whole class  $C$  of functions continuous in  $[-1, 1]$ . Bernstein showed that for every  $X$ , there exists a function  $f_0(x)$  and a point  $x_0$  in  $C[-1, 1]$  such that

$$(1.1.4) \quad \overline{\lim}_{n \rightarrow \infty} L_n[f_0, x_0; X] = \infty.$$

L. Fejér [1916] showed that if instead of Lagrange interpolation, we consider the Hermite-Fejér interpolation polynomials, the situation changes. The Hermite-Fejér polynomials  $H_{n+1}[f, x, X]$  are of degree  $\leq 2n + 1$  and are uniquely determined by

(1.1.5)  $H_{n+1}[f, x_{kn}; X] = f(x_{kn})$ ,  $H_{n+1}^{(1)}[f, x_{kn}; X] = \delta_{kn}$   
 where  $\delta_{kn}$  are arbitrary real numbers,  $k = 0, 1, \dots, n$ .  
 The explicit form of  $H_{n+1}[f, x; X]$  is given by

$$(1.1.6) \quad H_{n+1}[f, x; X] = \sum_{k=0}^n f(x_{kn}) h_{kn}(x) + \sum_{k=0}^n \delta_{kn} l_{kn}(x)$$

where

$$(1.1.7) \quad h_{kn}(x) = \left\{ 1 - \frac{\omega_n''(x_{kn})(x - x_{kn})}{\omega_n'(x_{kn})} \right\} l_{kn}^2(x) \\ =: v_{kn}(x) l_{kn}^2(x)$$

and

$$(1.1.8) \quad \eta_{kn}(x) = (x - x_{kn}) l_{kn}^2(x).$$

Fejér brought out the importance of Hermite interpolation by introducing the concept of "strongly normal" point systems. To each set of  $n + 1$  distinct points  $x_0, x_1, \dots, x_n$ , Fejér associates a set of  $n + 1$  points  $X_0, X_1, \dots, X_n$  which are the zeros of the linear functions  $k(x)$ . The points  $X_0, X_1, \dots, X_n$  are said to be the conjugate point system of  $x_0, x_1, \dots, x_n$ . A system of points  $x_0, x_1, \dots, x_n$  is called strongly normal if the conjugate point system lies inside  $[-1, 1]$ . For example, the zeros of the Tchebycheff polynomial  $T_n(x) = \cos n\theta$ ,  $\cos \theta = x$  form a strongly normal point system. Fejér proved (using these ideas) that Hermite-Fejér interpolation polynomials based on strongly normal point systems (and under certain conditions on  $\delta_{kn}$ ) converge uniformly to  $f(x)$  on  $[-1, 1]$ .

Optimal Error Bounds for Two Point Hermite Interpolation

In order to motivate the present day work on error bounds, we first consider the classic error bound of Cauchy. Let us consider once more the interpolation formula of Lagrange. Let  $f(x) \in C[a,b]$  and consider the Lagrange interpolation polynomial

$$L_n[f, x] = \sum_{k=0}^n f(x_{kn}) l_{kn}(x) .$$

Next we set

$$(1.2.1) \quad e(x) = f(x) - L_n[f, x] .$$

In the case  $f(x)$  is itself a polynomial of degree  $\leq n$ , then it is easy to see from the uniqueness of the Lagrange interpolation polynomial that  $e(x) = 0$ . Thus it is of interest to study what can be said about  $e(x)$  if  $f(x)$  is a given smooth function other than a polynomial of degree  $\leq n$ . The following theorem gives the most widely known error bound.

Theorem 1.1 (Cauchy). Let  $f(x) \in C[a,b]$  and suppose that  $f^{(n)}(x)$  exists at each point of  $[a,b]$ . Let  $L_n[f, x]$  be the element of the class of polynomials of degree  $\leq n - 1$  that satisfies the equation

$$(1.2.2) \quad L_n[f, x_{in}] = f(x_{in}) \quad i = 0, 1, \dots, n.$$

Then for any  $x$  in  $[a,b]$ , the error

$$e(x) = f(x) - L_n[f, x]$$

has the value

$$(1.2.3) \quad e(x) = \omega_n(x) f^{(n+1)}(\xi) / (n+1)! ,$$

where  $\xi$  is a point of  $[a,b]$  that depends on  $x$  and

$$\omega_n(x) = \sum_{i=0}^n (x-x_{in}).$$

An immediate consequence of (1.2.3) is the inequality

$$(1.2.4) \quad |e(x)| \leq |\omega_n(x)| \|f^{(n+1)}\|/(n+1)!$$

where  $\| \cdot \|$  denotes the supremum norm on  $[a,b]$ . If we set  $f(x) = \omega_n(x)$ , we see that (1.2.4) becomes an equality. Thus the right hand side cannot be made smaller. We therefore say that (1.2.4) is an optimal bound.

The Equations (1.2.3) and (1.2.4) have been extensively studied. For instance, the study of minimizing  $\|\omega_n\|$  led to Tchebychev's system of orthogonal polynomials. For a good discussion of some of the elementary analysis associated with this error bound, see Powell [1981].

In contrast to the precise and beautiful pointwise Cauchy bound, very little has been known about precise polynomial derivative errors. Denoting  $e(x)$  as the Cauchy remainder for Lagrange polynomial interpolation, we consider the role played by the term  $f^{(n+1)}(\xi)$ . If  $f \in P_n$  (the class of polynomials of degree  $\leq n$ ), the remainder vanishes identically. For a fixed  $x$ , we may consider the remainder

$$e_n(x) = f(x) - L_n[f,x]$$

as a process which annihilates all elements of  $P_n$ . We may now formulate the following theorem of Peano [1913].

Theorem 1.2 (Peano). Let  $L$  be a continuous linear functional such that  $L(p) = 0$  for all  $p \in P_n$ . Then for all  $f \in C^{(n+1)}[a,b]$ ,

$$(1.2.4) \quad L(f) = \int_a^b f^{(n+1)}(t) K(t) dt$$

where

$$K(t) = \{ L_x[(x-t)_+^n] \} / n!$$

and

$$\begin{aligned} (x-t)_+^n &= (x-t)^n && \text{for } x \geq t \\ &= 0 && \text{for } x < t. \end{aligned}$$

The notation  $L_x[(x-t)_+^n]$  means that the functional  $L$  is applied to  $(x-t)_+^n$  considered as a function of  $x$ . For a detailed study of the Peano theorem we refer to P. J. Davis [1975] and to A. Sard [1963]. We next turn to an application of the Peano theorem to derive pointwise optimal derivative error bounds.

Let  $u(x) \in C^{(4)}[0, h]$  be given; let  $v_3(x)$  be the unique Hermite interpolation polynomial of degree  $\leq 3$  satisfying

$$\begin{aligned} (1.2.5) \quad v_3(0) &= u(0), & v_3(h) &, \\ v'_3(0) &= u'(0), & v'_3(h) &= u'(h). \end{aligned}$$

Ciarlet, Schultz and Varga [1967] obtained a pointwise error bound for  $e(x) = v_3(x) - u(x)$  and its derivatives in terms of

$$U = \max_{0 \leq x \leq h} |u^{(4)}(x)|.$$

Their bounds are

$$(1.2.6) \quad |e^{(k)}(x)| \leq \frac{h^k [x(h-x)]^{2-k}}{k! (4-2k)!} U \quad k = 0, 1, 2.$$

For  $k = 0$ , (1.2.6) is best possible, since equality holds for  $u(x) = x^2(h-x)^2$ , whose Hermite interpolation polynomial is  $v=0$ .

G. Birkhoff and A. Priver [1967] obtained the following optimal error bounds on the derivative  $|e^{(k)}(x)|$  in terms of  $U$ .

Theorem 1.3 (Birkhoff and Priver). Let  $u(x) \in C^4[0,1]$ .

Then we have ( $h = 1$ )

$$(1.2.7) \quad |e'(x)|/U \leq [x(x-1)(2x-1)] / 12$$

$$\text{for } 0 \leq x \leq 1/3,$$

$$\leq [16x^3 - 105x^2 + 197x - 162$$

$$+ 66/x - 13/x^2 + 1/x^3] / 96$$

$$\text{for } 1/3 \leq x \leq 1/2.$$

$$(1.2.8) \quad |e''(x)|/U \leq [48x^5 + 42x^4 - 100x^3$$

$$+ 54x^2 - 12x + 1] / 2(1-x)^3$$

$$\text{for } 0 \leq x \leq 1/3,$$

$$\leq [-6(x-1/2)^2 + 1/2] / 12$$

$$\text{for } 1/3 \leq x \leq 2/3.$$

$$(1.2.9) \quad |e'''(x)|/U \leq -(x-1/2)^4 + 3(x-1/2)^2/2 + 3/16$$

$$\text{for } 0 \leq x \leq 1.$$

For  $1/2 \leq x \leq 1$  the bounds of  $e^{(k)}(x)$  are given by

$$(1.2.10) \quad e^{(k)}(x) = e^{(k)}(1-x), \quad k = 0, 1, 2, 3.$$

Further, from Birkhoff & Priver, the uniform error bounds are given by

$$\begin{aligned}
 |e^{(r)}(x)| &\leq a_r U, \quad r = 1, 2, 3, \\
 a_0 &= \frac{1}{4^{24}!} \\
 (1.2.11) \quad a_1 &= (\sqrt{3})/216 \\
 a_2 &= 1/12 \\
 a_3 &= 1/2.
 \end{aligned}$$

The proof of the above theorem is based on the Peano kernel theorem. It gives a general and highly useful method for expressing the errors of approximations in terms of derivatives of the underlying functions of the approximation. For a computer routine which gives polynomial error bounds by numerical quadrature of the Peano kernel, see Howell and Diaa [1986]. Stroud [1974] gives a readable account of some other applications.

### Birkhoff Interpolation

We have just observed that in problems of Hermite interpolation, function values and consecutive derivatives are prescribed for given points. In 1906, G. D. Birkhoff considered those interpolation problems in which the consecutive derivative requirement can be dropped. This more general kind of interpolation is now referred to as the Birkhoff (or the lacunary) interpolation problem(s).

The Birkhoff interpolation problem differs from the more familiar Lagrange and Hermite interpolation in both its problems and its methods. For example, Lagrange and Hermite interpolation problems are always uniquely

solvable for every choice of nodes, but a given Birkhoff interpolation may not give a unique solution.

More formally, given  $n + 1$  integer pairs  $(i, k)$  corresponding to  $n + 1$  real numbers  $c_{i,k}$ , and  $m$  distinct real numbers  $x_i$ ,  $i = 1, 2, \dots, m \leq n + 1$ , a given problem of polynomial interpolation is to satisfy the  $n + 1$  equations

$$(1.3.1) \quad P_n^{(k)}(x_i) = y_{i,k}$$

with a polynomial  $P_n$  of degree at most  $n$ . (We are using the convention that  $P_n^{(0)}(x) = P_n(x)$ .)

If for each  $i$ , the orders  $k$  of the derivatives in (1.3.1) form an unbroken sequence  $k = 0, 1, \dots, k_i$ , then the interpolation polynomial always exists, is unique, and can be given by an explicit formula. If some of the sequences are broken, we have Birkhoff interpolation. As remarked by Professor Lorentz [1983], the two cases are as different as, let us say, the theory of linear and nonlinear differential equations.

Pairs  $(i, k)$  which appear in (1.3.1) are most easily described by means of the interpolation or incidence matrix  $E$ . If  $P_n^{(k)}(x_i)$  is specified in (1.3.1), we put a "1" in the  $i+1$ st column and  $k$ th row of  $E$ . If  $P_n^{(k)}(x_i)$  is not specified in (1.3.1), then a "0" appears in the  $i+1$ st column and  $k$ th row. Each of the  $m$  rows of  $E$  has a non-zero entry. An incidence matrix  $E$  and a pointset  $X$ , which lists the points  $x_i$ , specify a Birkhoff interpolation problem of the type of (1.3.1). For a given  $E$  and  $X$ , the

unique existence of an interpolation polynomial of degree  $n + 1$  is equivalent to the invertibility of the system of equations given by (1.3.1), or equivalently to the invertibility of a matrix  $V$  which we will refer to as a generalized Vandermonde matrix  $V$ . For Lagrange interpolation of the points  $x_i$ ,  $i = 1, 2, \dots, n + 1$ , the Vandermonde  $V$  is given as

$$(1.3.2) \quad V = \begin{matrix} & 1 & 1 & . & . & . & 1 \\ & x_1 & x_2 & . & . & . & x_{n+1} \\ & . & . & . & . & . & . \\ & x_1^n & x_2^n & . & . & . & x_{n+1}^n \end{matrix}$$

Inversion of the Vandermonde gives the coefficients of the fundamental functions  $l_{kn}(x)$  of Lagrange interpolation. As Lagrange interpolations are always unique, it follows that Vandermonde matrices are invertible.

For a given system (1.3.1), it is not hard to construct an analagous matrix to (1.3.2), which we will refer to as the generalized Vandermonde. Just as inverting the Vandermonde matrix gives the fundamental functions of Lagrange interpolation, inverting the generalized Vandermonde gives a convenient form for representing a Birkhoff interpolation. The Vandermonde and its counterpart for Birkhoff interpolation are examples of Gram matrices, of which a good account is to be found in Davis [1975].

Though invertible, the Vandermonde matrices are known to be extremely ill-conditioned for real-valued

interpolation. Many of the generalized Vandermonde matrices associated with Birkhoff interpolation processes are much better conditioned, illustrating an advantage of Birkhoff interpolation over the more traditional Lagrange interpolation. To make this point more explicit, we define "condition" of a matrix.

For a given norm  $|| \cdot ||$ , and invertible matrix  $M$ , we define the condition  $\text{cond}(M)$  of the matrix  $M$  by

$$(1.3.3) \quad \text{cond}(M) = ||M|| \, ||M^{-1}||.$$

If we rescale the Birkhoff interpolation problem specified by  $E$  and  $X$  to the unit interval, we can define the condition of an interpolation as the condition of the associated generalized Vandermonde. In the  $L_2$  norm for eleven equally spaced points, the condition number of Lagrangian interpolation is on the order of a million. On the other hand, Lagrangian interpolation on eleven equally spaced complex roots of unity has  $L_2$  condition number one, as does the eleven term MacLaurin expansion.

Computationally speaking, the inverse of the condition number of a matrix  $M$  is the norm distance of  $M$  from a singular matrix (See Golub and Van Loan [1983]). For example, the Vandermonde for Lagrange interpolation of eleven points on the unit interval is thus seen to be a norm distance of only one-millionth from being singular. Not only is the ill-conditionedness of the Vandermonde troublesome in determining the coefficients of the fundamental functions, but it also causes problems of

round-off error in evaluating a polynomial by use of the fundamental functions. For these reasons, it is very much preferable to use a well-conditioned interpolation.

The MacLaurin expansion, having diagonal generalized Vandermonde, is as well-conditioned as is possible. Another particularly well-conditioned interpolation is the Lidstone interpolation.

A Lidstone polynomial is a truncation of a Lidstone series. In turn, a Lidstone series is a generalization of a Taylor series which approximates a given function in the neighborhood of two points instead of one. Such series have been studied by G. J. Lidstone [1930], by Widder [1942], by Whittaker [1934] and by others. More precisely, the series has the form

$$(1.3.3) \quad f(x) = f(1)\Lambda_0(x) + f(0)\Lambda_0(1-x) + f''(1)\Lambda_1(x) + f''(0)\Lambda_1(1-x) + \dots$$

where  $\Lambda_n(x)$  is a polynomial of degree  $2n + 1$  defined by the relations

$$(1.3.4) \quad \begin{aligned} \Lambda_n(x) &= x \\ \Lambda_n''(x) &= \Lambda_{n-1}(x) \\ \Lambda_n(0) &= \Lambda_n(1) = 0, \quad n = 1, 2, \dots \end{aligned}$$

Thus it is clear that the sum of an even number of terms of the series (1.3.3) is a polynomial which coincides with  $f(x)$  at  $x = 0$  and at  $x = 1$ . Moreover, each even derivative of the polynomial coincides with the corresponding derivative of  $f(x)$  at those points.

### Polynomial Approximation

Weierstrass first enunciated the theorem that an arbitrary continuous function can be approximately represented by a polynomial with any degree of accuracy. We may express this theorem in the following form.

If  $f(x)$  is a given function, continuous for  $a \leq x \leq b$ , and if  $\epsilon$  is a given positive quantity, it is always possible to define a polynomial  $P(x)$  such that

$$(1.4.1) \quad |f(x) - P(x)| < \epsilon$$

for all  $a \leq x \leq b$ .

It is readily seen that the number of terms required to yield a specified degree of approximation, or under the converse aspect, the degree of approximation attainable with a specified number of terms, is related to the properties of continuity of  $f(x)$ . Naturally this has led to many interesting developments in the theory of degree of approximation of continuous functions by polynomials to which we turn to describe.

A first important step in building this theory was made by D. Jackson [1930]. Let  $f \in C[-1,1]$ . Suppose that we define the best approximation of  $f$  by polynomials of degree  $n$  by

$$(1.4.2) \quad E_n(f) = \inf ||f - P_n||$$

where  $P_n$  ranges over all algebraic polynomials of degree  $n$  and  $||f|| = \max |f(x)|$ ,  $a \leq x \leq b$ . Jackson considered the problem of estimating  $E_n(f)$ . To describe his results we need the following definition.

Definition 1.1 If  $f \in C[a,b]$ , then the modulus of continuity of  $f$  is a function  $(f,h)$  such that

$$(1.4.3) \quad (f,h) = \sup_{|x-y| \leq h; x,y \in [a,b]} |f(x) - f(y)|.$$

Now Jackson's theorems may be easily stated.

Theorem 1.4 (Jackson). Let  $f$  be continuous on  $[-1,1]$ . There is a positive constant  $A$  such that

$$(1.4.4) \quad E_n(f) \leq A \omega(f, 1/n), \quad n = 1, 2, \dots$$

where  $A$  is independent of  $f$ .

An important corollary of Theorem 1.3 deserves to be mentioned. Let  $\text{Lip}\alpha_{[-1,1]}(M)$  (or simply  $\text{Lip}\alpha$ ) be the class of functions  $f$  in  $C[-1,1]$  such that

$$|f(x) - f(y)| \leq M |x-y|^\alpha$$

for all  $x$  and  $y$  in  $[-1,1]$ . It is easy to see that

$f \in \text{Lip}\alpha_{[-1,1]}(M)$  if and only if

$$\omega(f,h) \leq M h^\alpha \text{ for all } h \geq 0.$$

We then have the following consequence of Jackson's theorem.

Corollary 1.5 Let  $0 < \alpha \leq 1$ . If  $f \in \text{Lip}\alpha_{[-1,1]}(M)$ , for some constant  $M$ , then

$$(1.4.5) \quad E_n(f) \leq \frac{A}{n^\alpha} \quad \text{for } n = 1, 2, \dots$$

for some positive constant  $A$ .

A. F. Timan [1951] noticed the following strengthening of Jackson's theorem.

Theorem 1.6 (Timan). There is a positive constant  $C$  such that if  $f \in C[-1,1]$  and  $n$  is a natural number, then there

is a polynomial  $P_n$  of degree  $n$  such that

$$(1.4.6) \quad |f(x) - P_n(x)| \leq A \left[ \omega\left(f, \frac{\sqrt{1-x^2}}{n}\right) + \omega\left(f, 1/n^2\right) \right]$$

for all  $x$  in the interval  $[-1,1]$ .

In this result, in contrast to the theorem of Jackson, the position of the point  $x$  in the interval  $[-1,1]$  is taken into consideration and it is apparent that for the polynomial  $P_n(x)$  thus constructed, as  $|x| \rightarrow 1$ , the deviation  $|f(x) - P_n(x)|$  is of magnitude  $\omega(f, 1/n^2)$ .

Following the important theorem of Timan, V. K. Dzjadyk [1956] proved the converse of Jackson's theorem.

Theorem 1.7 (V. K. Dzjadyk). Let  $f \in C[-1,1]$ . Suppose that  $0 < \alpha < 1$ . Then there is a constant  $B$  such that to each  $n$  there corresponds a polynomial  $P_n$  of degree  $n$  such that

$$(1.4.7) \quad |f(x) - P_n(x)| \leq B \left\{ \left( \frac{\sqrt{1-x^2}}{n} \right)^\alpha + \left( \frac{1}{n^2} \right)^\alpha \right\}$$

if and only if  $\omega(f, h) \leq C h^\alpha$  for some constant  $C$ .

From Jackson's theorem we noticed that if  $f \in \text{Lip}\alpha$ , then

$$E_n(f) \leq \frac{AM}{n^\alpha}, \quad n = 1, 2, \dots$$

where  $A$  is an absolute constant. To achieve a more rapid decrease to 0 of  $E_n(f)$ , it is necessary to assume more smoothness for  $f$ , for example, that  $f$  has several continuous derivatives. Let  $C^r[-1,1]$ ,  $r = 0, 1, \dots$  denote the subset of  $C[-1,1]$  consisting of those functions

which possess  $r$  continuous derivatives on  $[-1,1]$ . For this class of functions, Dunham Jackson proved also the following direct theorem.

Theorem 1.8 (D. Jackson). If  $f \in C^{(r)}[-1,1]$ , then

$$(1.4.8) \quad E_n(f) \leq A_r (1/n)^r \omega(f^{(r)}, 1/n), \quad n = 1, 2, \dots$$

For many important contributions we refer to the work of G. G. Lorentz [1983].

### Spline Approximation

One uses polynomials for approximation because they can be evaluated, differentiated and integrated easily and in finitely many steps using just the basic arithmetic operations of addition, subtraction and multiplication. But there are limitations of polynomial approximations. For example, the polynomial interpolant is very sensitive to the choice of interpolation points. If the function to be approximated is badly behaved anywhere in the interval of approximation, then the approximation is poor everywhere.

This global dependence on local properties can be avoided when using piecewise polynomial approximation. Concerning piecewise polynomial approximation, Professor I. J. Schoenberg remarked that "polynomials are wonderful even after they are cut into pieces, but the cutting must be done with care. One way of doing the cutting leads to the so-called spline functions" (Schoenberg [1946], p. 46).

Splines were introduced by Prof. Schoenberg in 1946 as a tool for the approximation of functions. They tend to be smoother than polynomials and to provide better approximation of low order derivatives. Though we will later use the word spline in a somewhat broader context, we first give the more traditional definition.

Let

$$(1.5.1) \quad x_1 < x_2 < \dots < x_k$$

be a sequence of strictly increasing real numbers called the knots of the spline function. We may say  $s_m(x)$  is a spline function of degree  $m$  having the knots

$$x_1, x_2, \dots, x_k$$

if it satisfies

$$a) \quad s_m(x) \in C^{m-1}(-\infty, \infty);$$

b) In each interval  $(x_i, x_{i+1})$ , including  $(-\infty, x_1)$  and  $(x_k, \infty)$ , the restriction of  $s_m(x)$  to  $(x_i, x_{i+1})$  is a polynomial of degree at most  $m$ . Thus, a step function  $s_0(x)$  may be regarded as a spline function of degree 0, while a spline function of degree 1 is a polygon (broken line function) with possible corners at some or all of the possible corners at some or all of the points (1.5.1). Similarly,  $s_2(x)$  has a graph composed of a sequence of parabolas which join at the knots continuously together with their slopes. Both for a smoother approximation and for a more efficient approximation, one has to go to piecewise polynomial approximation with higher order pieces. The most popular choice continues to be a

piecewise cubic approximating function. Various kinds of cubic splines are in use in numerical analysis. The ones most commonly used are complete cubic splines, periodic cubic splines and natural cubic splines.

A spline function of degree  $m$  with  $k$  knots is represented by a different polynomial in each of the  $k+1$  intervals into which the  $k$  knots divide the real line. As each polynomial involves  $m + 1$  parameters, the spline function involves a total of  $(m+1)(k+1)$  parameters. However, the continuity conditions stated earlier impose certain constraints on those parameters. At each knot, the two adjoining polynomial arcs must have equal ordinates and equal derivatives of order  $1, 2, \dots, m - 1$ . Thus,  $m$  constraints are imposed. It is easy to see that every spline function  $s(x)$  of degree  $m$  with the knots  $x_1, x_2, \dots, x_k$  has a unique representation in the form

$$(1.5.1) \quad s(x) = P_m(x) + \sum_{j=1}^k c_j (x - x_j)_+^m$$

where  $P_m(x)$  denotes a polynomial of degree  $m$  and

$$(1.5.2) \quad \begin{aligned} x_+^m &= x^m & x &\geq 0 \\ &= 0 & x &< 0. \end{aligned}$$

Also

$$(1.5.3) \quad c_j = (1/(m!)) \{ S^{(m)}(x_{j+}) - S^{(m)}(x_{j-}) \}.$$

The class of "natural" spline functions was introduced by Prof. Schoenberg [1946]. A spline function  $s(x)$

of odd degree  $2p-1$  with knots  $x_1, x_2, \dots, x_k$  is called a natural spline function if the two polynomials by which it is represented in the two end intervals  $(-\infty, x_1)$  and  $(x_k, +\infty)$  are of degree  $p-1$  or less. It is easy to express the natural spline functions by

$$(1.5.4) \quad s(x) = P_{p-1}(x) + \sum_{j=1}^k C_j (x-x_j)_+^{2p-1}$$

where

$$\sum_{j=1}^k C_j x_j^r = 0, \quad r = p, p+1, \dots, 2p-1.$$

The following theorem states an important interpolation property of natural spline functions.

Theorem 1.9 Let  $(x_i, y_i)$ ,  $i = 1, 2, \dots, k$ , be given data points, where the  $x_i$ 's form a strictly increasing sequence, and let  $p$  be a positive integer not exceeding  $n$ . Then there is a unique natural spline function  $s(x)$  of degree  $2p - 1$  with the knots  $x_i$  such that

$$(1.5.5) \quad s(x_i) = y_i, \quad i = 1, 2, \dots, k.$$

Natural spline functions possess certain impressive optimal properties and can be shown to be the "best" approximating functions in a certain sense. This is the content of the next theorem.

Theorem 1.10 Let  $P(x)$  be the unique natural spline function that interpolates the data points  $(x_i, y_i)$ ,  $i = 1, 2, \dots, k$ , in accordance with Theorem 1.7. Let  $f(x)$  be any function of the class  $C^{(p)}$  that satisfies the conditions

$$(1.5.6) \quad f(x_i) = y_i, \quad i = 1, 2, \dots, k.$$

Let  $(a, b)$  be a finite interval containing all the knots  $x_i$ . Then

$$(1.5.7) \quad \int_a^b [f^{(p)}(x)]^2 dx \geq \int_a^b [s^{(p)}(x)]^2 dx$$

with equality only if  $f(x) = s(x)$ .

The effectiveness of the spline approximation can be explained to a considerable extent by its striking convergence properties. Interesting contributions were made by J. N. Ahlberg and E. N. Nilson [1964], C. DeBoor and G. Birkhoff [1964], A. Sharma and A. Meir [1967], M. J. Marsden [1972], T. R. Lucas [1974], E. W. Cheney and F. Schurer [1968], C. A. Hall [1968], C. A. Hall and W. W. Meyer [1976], and A. K. E. Atkinson [1968]. As a good reference on splines which offers a good comparison of the approximating properties of polynomials and splines, we recommend A Practical Guide to Splines by C. DeBoor [1978].

First we discuss error analysis for the class of functions  $f(x) \in C^{(2)}$  with period one. Let

$$(1.5.8) \quad \{x_i\}_{i=0}^k: 0 = x_{n0} < x_{n1} < \dots < x_{n, k_n} = 1$$

be a division of  $[0, 1]$  of mesh gauge

$$(1.5.9) \quad h_n = \max_{0 \leq i \leq k_n} (h_{n,i}),$$

where

$$h_{n,i} = x_{n,i} - x_{n,i-1}.$$

A periodic cubic spline function  $y_n(x)$  is a function composed of a cubic polynomial in each of the intervals of  $\{x_i\}_{i=0}^k$  with the requirement that

$$y_n(x) \in C^{(2)}[0,1]$$

and

$$y_n^{(i)}(0) = y_n^{(i)}(1), \quad i = 0, 1, 2.$$

It was observed by Walsh, Ahlberg and Nilson [1962] that there exists a unique periodic spline function  $y_n(x)$  which interpolates  $f(x)$  at the points  $x_{n,1}$ . It was shown that  $y_n(x)$  and  $y'_n(x)$  converge uniformly to  $f(x)$  and  $f'(x)$  respectively as  $h_n \rightarrow 0$ . Later Ahlberg and Nilson [1966] studied the more delicate question of the convergence of  $y''_n(x)$  to  $f''(x)$ . Writing

$$(1.5.10) \quad \lambda_{n,i} = h_{n,i+1} / (h_{n,i} + h_{n,i+1}), \\ i = 1, 2, \dots, k_n - 1$$

and

$$\Lambda_n = \max_{0 \leq i \leq k} |\lambda_{n,i} - 1/2|,$$

where for  $m = k_n$ ,  $\lambda_{n,m+1}$  is taken as

$$(h_{n,1} + h_{n,m}) / h_{n,1},$$

they show that

$$y''_n(x) \rightarrow f''(x)$$

uniformly provided that

$$h_n \rightarrow 0 \text{ and } \Lambda_n \rightarrow 0.$$

After this result, I. J. Schoenberg [1964a] raised the question that it would be very interesting to find out to what extent the condition  $\Lambda_n \rightarrow 0$  is really necessary in the above mentioned theorem. The above theorem together

with the open problem of Schoenberg lead to important contributions by Birkhoff and DeBoor [1964], and Meir and Sharma [1969] which we turn to describe.

In 1964, Garrett Birkhoff and Carl DeBoor made the following contribution. Let  $f(x) \in C'[0,1]$  and let

$$(1.5.11) \quad \{x_i\}_{i=0}^k, \quad 0 = x_0 < x_1 < \dots < x_n = 1$$

be a partition. The function  $f(x)$  is now interpolated by a cubic spline function  $s(x)$  (called a complete cubic interpolation spline function) which means that  $s(x)$  is a cubic polynomial when restricted to each interval  $(x_i, x_{i+1})$ , and  $s(x) \in C^{(2)}[0,1]$ . Moreover  $s(x)$  is uniquely defined by the conditions

$$(1.5.13) \quad \begin{aligned} f(x_i) &= s(x_i) \quad , \quad i = 0, 1, \dots, k ; \\ f'(0) &= s'(0) \quad , \\ f'(1) &= s'(1) \quad . \end{aligned}$$

This first important result concerning the error analysis yielded the following theorem.

Theorem 1.11 Let  $f(x) \in C^{(4)}[0,1]$ . Denote

$$e(r) = f(r) - s(r) \quad .$$

There are constants  $c_r(m)$ ,  $r = 0, 1, 2, 3$ , depending only on  $m > 0$ , such that

$$(1.5.13) \quad |e^{(r)}(x)| \leq c_r(m) h^{4-r} ||f^{(r)}|| \quad , \\ r = 0, 1, 2, 3,$$

provided that

$$\begin{aligned} m_h &\leq m \quad , \\ h_i &= x_{i+1} - x_i \quad , \end{aligned}$$

$$h = \max h_i ,$$

$$m_h = [\max(h_i)] / [\min(h_i)]$$

and  $\| \cdot \|$  denotes the supremum norm.

The authors go a step further and prove a convergence theorem related to  $f \in C^{(3)}[0,1]$ .

Theorem 1.12 Let  $f'''(x)$  be absolutely continuous on  $[0,1]$ . Let  $\{x_i\}_{i=0}^k$  (where  $k$  depends on  $n$ ) be a sequence of partitions of  $[0,1]$  such that  $h_n = \max_i h_{i,n} \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $m_{h,n} \leq m$  as  $n \rightarrow \infty$ . Let  $e_n(x)$  be the error incurred when  $f(x)$  is interpolated by a spline function on  $\{x_i\}_{i=0}^k$ . Then

$$|e'''_n| \rightarrow 0$$

uniformly on  $[0,1]$  as  $n \rightarrow \infty$ .

The next important development came with some interesting results by Prof. A. Sharma and A. Meir [1967] concerning degree of approximation of spline interpolation. This paper does away with some annoying assumptions under which uniform convergence of the interpolating cubic spline and its derivatives was proven earlier (see above for these restrictions).

Theorem 1.13 Let  $f(x)$  be continuous and periodic with period unity. Let

$$(1.5.15) \quad q_n = \max_{i,j} (h_{n,i}/h_{n,j})$$

where

$$h_{n,i} = x_{n,i+1} - x_{n,i} .$$

Let  $s_n(x)$  be the cubic spline of period unity with joints (or knots)  $x_{n,i}$ ,  $i = 0, 1, \dots, n$  in  $[0,1]$ , such

that  $s_n(x)$  interpolates  $f(x)$  at the joints. Let

$$||g|| = \max_x |g(x)| \quad \text{for } g \in C[0,1],$$

and

$$\omega(g, h) = \max \{ |g(u) - g(v)| : |u-v| \leq h \}, \quad h \geq 0.$$

The authors prove

i)

$$(1.5.16) \quad ||f - s_n|| \leq (1 + q_n^2) \omega(f, h_n);$$

ii) if  $f \in C^{(1)}$ , then

$$(1.5.17) \quad ||f^{(r)} - s_n^{(r)}|| \leq 76 h_n^{1-r} \omega(f', h_n),$$

$$r = 0, 1;$$

iii) if  $f \in C^{(2)}$ , then

$$(1.5.18) \quad ||f^{(r)} - s_n^{(r)}|| \leq 5 h_n^{2-r} \omega(f'', h_n),$$

$$r = 0, 1, 2;$$

iv) if  $f \in C^{(3)}$ , then

$$(1.5.19) \quad ||f^{(r)} - s_n^{(r)}|| \leq C h_n^{3-r} \omega(f''', h_n),$$

$$r = 0, 1, 2, 3;$$

where

$$C = 1 + q_n(1+q_n)^2$$

or

$$C = 1 + (1 + p_n)^2 / (2 - p_n)$$

with

$$p_n = \max_i (h_{n,i}/h_{n,j}) \quad \text{for } j = i-1, i+1$$

satisfying

$$p_n < 2.$$

From these results one can draw the obvious conclusions regarding uniform convergence of the interpolating

splines and derivatives. The arguments are surprisingly simple. The uniform convergence of  $s_n''$  to  $f''$ , which follows from iii), had been proved earlier by Ahlberg and Nilson (see above) under the additional assumptions that the mesh become eventually uniform, i.e.,

$$(1.5.20) \quad \lim_{n \rightarrow \infty} [h_{n,i}/(h_{n,i} + h_{n,i+1})] = 1/2 .$$

### Parabolic Spline Interpolation

Many interesting results were obtained by M. Marsden [1974] concerning the approximation of functions by even degree splines. Of particular interest are the simple parabolic splines. If break points are the same as the interpolated points, then the resulting spline is ill-behaved, as can be seen by simple examples (DeBoor [1978]). On the other hand, if we take the interpolated points midway between break points, the parabolic splines are very well-behaved. In fact in the first theorem given below, a good approximation to a continuous function is assured with no conditions on the partition other than the length of the largest subinterval being small.

We first give some necessary notation. Let

$$(1.6.1) \quad \{x_i\}_{i=0}^k: 0 = x_0 < x_1 < \dots < x_n = 1$$

be a fixed partition of  $[0,1]$ . Set

$$(1.6.2) \quad \begin{aligned} h_i &= x_i - x_{i-1} , & h &= \max_i h_i , \\ z_i &= (x_i + x_{i-1})/2 , \\ h_0 &= h_n , & a_i &= h_{i+1}/(h_i + h_{i+1}) , \\ c_i + a_i &= 1 , & \text{for } i &= 1, 2, \dots, n. \end{aligned}$$

Let

$$y \in C[0,1] , \quad y(0) = y(1) ,$$

$$||y|| = \sup \{ |y(x)| : 0 \leq x \leq 1 \}$$

such that  $y$  is extended periodically with period 1.

A function  $s(x)$  is defined to be a periodic quadratic spline interpolant associated with  $y$  and  $\{x_i\}_{i=0}^n$  if

(1.6.3) a)  $s(x)$  is a quadratic expression on each

$$(x_{i-1}, x_i) ,$$

$$b) s(x) \in C'[0,1] ,$$

$$c) s(0) = s(1) , \quad s'(0) = s'(1) ,$$

$$d) s(x_i) = y(x_i) , \quad i = 1, 2, \dots, n.$$

The following theorems were obtained by Marsden.

Theorem 1.14 (Marsden). Let  $\{x_i\}_{i=0}^k$  be a partition of  $[0,1]$ ,  $y(x)$  be a continuous 1- periodic function and  $s(x)$  be the periodic quadratic spline interpolant associated with  $y$  and  $\{x_i\}_{i=0}^n$ .

Then

$$(1.6.4) \quad ||s_i|| \leq 2 ||y|| , \quad ||s|| \leq 2 ||y|| ,$$

$$||e_i|| \leq 2 \omega(y, h/2) ,$$

$$||e|| \leq 3 \omega(y, h/2) .$$

(where  $s_i = s(x_i)$  and  $e_i = y(x_i) - s(x_i)$ ).

The constant 2 which appears in the first of the above equations can not, in general, be decreased.

Theorem 1.15 (Marsden). Let  $y$  and  $y'$  be continuous 1- periodic functions. Then

$$(1.6.5) \quad ||s'_i|| \leq 2 ||y'|| ,$$

$$\begin{aligned}
||e'_i|| &\leq 3 ||y'|| , \\
||e_i|| &\leq h \omega(y', h/2) , \\
||e_i|| &\leq h ||y'|| , \\
||e|| &\leq (5/4) h ||y'|| , \\
||e'_i|| &\leq 3 \omega(y', h/2) , \\
||e'|| &\leq (9/2) \omega(y', h/2) , \\
||e|| &\leq (13/8) h \omega(y', h/2) .
\end{aligned}$$

Theorem 1.16 (Marsden). Let  $y$ ,  $y'$ , and  $y''$  be continuous 1- periodic functions. Then

$$\begin{aligned}
(1.6.6) \quad ||e_i|| &\leq (1/8) h^2 \omega(y'', h) , \\
||e'_i|| &\leq (1/2) h \omega(y'', h) , \\
||e'|| &\leq 2 h ||y''|| , \\
||e|| &\leq (5/8) h^2 ||y''|| , \\
|e''(x)| &\leq [1 + (h/h_i)] \omega(y'', h) , \\
x_i &\leq x \leq x_{i+1} .
\end{aligned}$$

Theorem 1.17 (Marsden). Let  $y$ ,  $y'$ ,  $y''$ , and  $y'''$  be continuous 1- periodic functions. Then

$$\begin{aligned}
(1.6.7) \quad ||e|| &\leq (17/96) h^3 ||y'''|| , \\
||e'|| &\leq (11/24) h^2 ||y'''|| , \\
||e''|| &\leq [h_i + (2 h^2/3 h_i)] ||y'''|| , \\
x_i &< x < x_{i+1} .
\end{aligned}$$

### Optimal Error Bounds for Cubic Spline Interpolation

An interesting application of the theorem of Birkhoff and Priver [1967] (discussed above) was given by Hall [1968] and subsequently by Hall and Meyer [1976], concerning optimal error bounds for cubic spline interpolation.

In order to describe these results let  $f \in C^{(4)} [0,1]$  and let  $s(x)$  be the complete cubic spline function satisfying the conditions (1.5.13). The main result of Hall and Meyer may now be stated.

Theorem 1.18 (Hall and Meyer). Let  $s(x)$  be the unique complete cubic spline interpolation satisfying (1.5.13).

Suppose

$$f \in C^{(4)} [0,1].$$

Then for  $0 \leq x \leq 1$

$$(1.7.1) \quad |f^{(r)}(x) - s^{(r)}(x)| \leq c_r h^{4-r} ||f^{(4)}||$$

$$r = 0, 1, 2$$

with

$$h = \max(x_{i+1} - x_i), \quad c_0 = 5/384,$$

$$c_1 = 1/24, \quad c_2 = 3/8.$$

Further, the constants  $c_0$  and  $c_1$  are optimal in the sense that

$$(1.7.2) \quad c_r = \sup \frac{|| (f - s)^{(r)} ||}{h^{4-r} ||f^{(4)}||}$$

where the supremum is taken over all  $\{x_i\}_{i=0}^k$  partitioning  $[0,1]$  and over all  $f \in C^{(4)} [0,1]$  such that  $f^{(4)}$  is not identically equal to zero.

Varma and Katsifarakis (in press) were able to resolve the cases of  $f \in C^{(3)}$  and  $f \in C^{(2)}$  in the following theorems. Let  $s(x)$  be the unique complete cubic spline satisfying the relationship:

$$(1.7.3) \quad s(x_i) = f(x_i), \quad i = 0, 1, \dots, k;$$

$$s'(x_i) = f'(x_i), \quad i = 0, k.$$

Theorem 1.19 If  $f$ ,  $f'$ ,  $f''$ , and  $f'''$  are continuous on  $[0,1]$ , then

$$(1.7.4) \quad |s^{(r)}(x) - f^{(r)}(x)| \leq c_r h^{3-r} \max_{0 \leq x \leq 1} |f'''(x)|$$

$$r = 0, 1, 2$$

where

$$c_0 = 1/96 + 1/27, \quad c_1 = 4/27,$$

$$c_2 = 1/2 + 4/(3\sqrt{3}).$$

Theorem 1.20 If  $f, f'$ , and  $f''$  are continuous on  $[0,1]$ , then

$$(1.7.5) \quad |s^{(r)}(x) - f^{(r)}(x)| \leq a_r h^{2-r} \omega(f'', h)$$

where

$$a_0 = 13/48, \quad a_1 = 5/6, \quad a_2 = 4.$$

CHAPTER TWO  
BEST ERROR BOUNDS FOR DERIVATIVES OF  
TWO POINT LIDSTONE POLYNOMIALS

Introduction and Statement of Main Theorem

Let  $u \in C^{2m}[0, h]$  be given and let  $v_{2m-1}$  be the unique Hermite interpolation of degree  $2m - 1$  matching  $u$  and its first  $m-1$  derivatives  $u^{(j)}$  at 0 and  $h$ . Let  $e = v_{2m-1} - u$  be the error function. For the special cases  $m = 2$  and  $m = 3$ , G. Birkhoff and A. Priver [1967] obtained pointwise optimal error bounds on the derivatives  $e^{(k)}$ ,  $0 \leq k \leq 2m - 1$  in terms of  $h$  and  $\max_{0 \leq x \leq h} |u^{(2m)}(x)|$ . These results are described in detail in Chapter One. Birkhoff and Priver note that for the cases  $m > 3$ , their method is not likely to give analytically exact bounds, though it can be adapted to give numerical approximations to pointwise exact error bounds. In the next chapter, we will directly apply the results of Birkhoff and Priver to the case of  $u$  in  $C^{(2m)}[0, h]$  and the interpolatory polynomial  $w_{2m-1}$  which matches  $u$  at 0 and  $h$  and which also matches the 2nd through  $m$ th derivatives of  $u$  at 0 and  $h$ .

Analogously to using Hermite interpolation polynomials, one may choose to approximate a given function  $u(x)$  in  $C^{2m}[0, h]$  by the so-called Lidstone interpolation polynomial  $L_{2m-1}[u, x]$  of degree  $\leq 2m - 1$

matching  $u$  and its first  $m - 1$  derivatives  $u^{(2j)}$  at 0 and  $h$ . Thus  $L_{2m-1}[u, x]$  satisfies the following conditions (where we assume  $h = 1$ ):

$$\begin{aligned} (2.1.1) \quad L_{2m-1}^{(2p)}[u, 0] &= u^{(2p)}(0) , \\ L_{2m-1}^{(2p)}[u, 1] &= u^{(2p)}(1) , \\ p &= 0, 1, \dots, m-1. \end{aligned}$$

The explicit formula for  $L_{2m-1}[u, x]$  is

$$\begin{aligned} (2.1.2) \quad L_{2m-1}[u, x] &= \sum_{i=0}^{m-1} u^{(2i)}(1) \Lambda_i(x) \\ &\quad + \sum_{i=0}^{m-1} u^{(2i)}(0) \Lambda_i(1-x) \end{aligned}$$

where

$$(2.1.3) \quad \Lambda_i(x) = \frac{2^{2i}}{(2i+1)!} B_{2i+1} \frac{(1+x)}{2}, \text{ for } i \geq 1$$

and

$$(2.1.4) \quad \Lambda_0(x) = x.$$

Here  $B_n(x)$  denotes the Bernoulli polynomial

$$(2.1.5) \quad B_n(x) = \sum_{k=0}^n \binom{n}{k} x^k B_{n-k}$$

and where the constant  $B_j$  is given by

$$(2.1.6) \quad B_j = \sum_{k=0}^j \binom{j}{k} B_k, \quad B_0 = 1.$$

That (2.1.2) in fact satisfies (2.1.1) follows from the facts

$$\begin{aligned} (2.1.7) \quad \Lambda_i^{(2p)}(0) &= 0, \quad p = 0, 1, \dots, i; \\ \Lambda_i^{(2p)}(1) &= 0, \quad p = 0, 1, \dots, i-1; \\ \Lambda_i^{(2i)}(1) &= 1. \end{aligned}$$

The main object of this chapter is to obtain pointwise optimal error bounds for

$$e^{(j)}(x) = f^{(j)}(x) - L_{2m-1}^{(j)}[f, x]$$

in terms of  $U = \max_{0 \leq x \leq 1} |u^{(2m)}(x)|$ . Here  $L_{2m-1}^{(j)}[f, x]$  denotes the  $j$ th derivative of the Lidstone polynomial defined by (2.2.2). An important role in Theorem 2.1 (see below) is played by the Euler polynomial  $Q_{2m}(x)$  of degree  $2m$  given by the formula

$$(2.1.8) \quad Q_{2m}(x) = - \int_0^1 G_1(x, t) Q_{2m-2}(t) dt, \quad m = 1, 2, \dots$$

where

$$(2.1.9) \quad Q_0(x) = 1$$

and

$$(2.1.10) \quad G_1(x, t) = t(x-1), \quad 0 \leq t < x \leq 1 \\ = x(t-1), \quad 0 \leq x < t \leq 1.$$

We may now state the main theorem as follows.

Theorem 2.1. Let  $u(x) \in C^{2m}[0, 1]$  and let  $L_{2m-1}[u, x] = L_{2m-1}(x)$  be the unique polynomial of degree  $\leq 2m-1$  satisfying the conditions (2.1.1). Then, for  $0 \leq x \leq 1$ , with

$$(2.1.11) \quad U = \max_{0 \leq x \leq 1} |u^{(2m)}(x)|, \\ |u^{(2j)}(x) - L_{2m-1}^{(2j)}(x)| \leq U Q_{2m-2j}(x), \\ j = 0, 1, \dots, m-1 \\ \leq U Q_{2m-2j}(1/2), \\ j = 0, 1, \dots, m-1$$

and for  $j = 1, 2, \dots, m$

$$(2.1.12) \quad |u^{(2j-1)}(x) - L_{2m-1}^{(2j-1)}(x)|$$

$$\begin{aligned} &\leq U \{ (1-2x) Q_{2m+2-2j}'(x) \\ &\quad + 2Q_{2m+2-2j}(x) \} \\ &\leq U |Q_{2m+2-2j}'(0)| \end{aligned}$$

where for a given integer  $k$ ,  $Q_{2k}(x)$  is the well known Euler polynomial defined by (2.1.7). Moreover, (2.1.11) and (2.1.12) are both best possible in the sense that there exists a function  $u(x) \in C^{2m}[0,1]$  such that (2.1.11) and (2.1.12) become equality for every  $x \in [0,1]$ .

From (2.2.11) and (2.1.12) follow immediately the also exact bounds

$$(2.1.13) \quad ||u^{(2j)} - L_{2m-1}^{(2j)}|| \leq Q_{2m-2j}(1/2) ||u^{(2m)}||, \\ j = 0, 1, \dots, m-1$$

and

$$(2.1.14) \quad ||u^{(2j-1)} - L_{2m-1}^{(2j-1)}|| \\ \leq |Q_{2m+2-2j}'(0)| ||u^{(2m)}||, \\ j = 1, 2, \dots, m-1$$

where  $|| \quad ||$  denotes the supremum norm on  $[0, 1]$ .

### Preliminaries

It is well known that the Bernoulli polynomials defined by (2.1.5) satisfy

$$(2.2.1) \quad B_n'(x) = nB_{n-1}(x)$$

and

$$(2.2.2) \quad B_n(1-x) = (-1)^n B_n(x).$$

In particular it follows that

$$(2.2.3) \quad B_{2n+1}(1/2) = 0.$$

From (2.2.1), (2.2.3) and (2.1.3)-(2.1.6), we obtain

$$(2.2.4) \quad \Lambda_i''(x) = \Lambda_{i-1}(x), \quad \Lambda_i(0) = 0, \quad \Lambda_i(1) = 0, \\ i \geq 1.$$

The proof of Theorem 2.1 depends on repeated use of the kernel  $G_1(x,t)$  defined by (2.1.10). Let us consider

$$(2.2.5) \quad g(x) = \int_0^1 G_1(x,t) r(t) dt \\ = \int_0^x (x-1)t r(t) dt + \int_x^1 (t-1)x r(t) dt.$$

On differentiating, we have

$$g'(x) = \int_0^x t r(t) dt + (x-1)x r(x) \\ - \int_x^1 (t-1) r(t) dt - x(x-1) r(x) \\ = \int_0^x t r(t) dt + \int_x^1 (t-1) r(t) dt.$$

Differentiating once more with respect to  $x$  we obtain

$$(2.2.6) \quad g''(x) = x r(x) - (x-1) r(x) = r(x).$$

Also

$$(2.2.7) \quad g(0) = g(1) = 0.$$

Let  $r(t) = \Lambda_{m-1}(t)$  in (2.2.5). From the above discussion it follows that

$$g(x) = \int_0^1 G_1(x,t) \Lambda_{m-1}(t) dt$$

satisfies

$$(2.2.8) \quad g''(x) = \Lambda_{m-1}(x), \quad g(0) = g(1) = 0.$$

From (2.2.4) we also know that for  $i \geq 1$

$$\Lambda_i''(x) = \Lambda_{i-1}(x), \quad \Lambda_i(0) = 0, \quad \Lambda_i(1) = 0.$$

Therefore

$$(2.2.9) \quad g(x) = \Lambda_m(x) = \int_0^1 G_1(x,t) \Lambda_{m-1}(t) dt.$$

From (2.1.9) it follows that

$$(2.1.10) \quad G_1(x,t) \leq 0.$$

Also  $\Lambda_0(t) = t \geq 0$ ,  $0 \leq t \leq 1$ . Therefore we obtain from (2.2.9) that

$$(2.2.11) \quad \Lambda_1(x) \leq 0, \quad 0 \leq x \leq 1.$$

On using (2.2.9), (2.2.10), and (2.2.11), we can assert that

$$(2.2.12) \quad \Lambda_2(x) \geq 0, \quad 0 \leq x \leq 1.$$

Inductively, it follows that  $\Lambda_m(x) \geq 0$  for  $0 \leq x \leq 1$  provided  $m$  is an even positive integer and  $\Lambda_m(x) \leq 0$ ,  $0 \leq x \leq 1$  if  $m$  is an odd positive integer. This property of  $\Lambda_m(x)$  will be needed many times in the proof of the theorem.

The following iteratively defined kernels comprise the essential machinery of the proof. Define

$$(2.2.13) \quad G_2(x,t) = \int_0^1 G_1(x,y) G_1(y,t) dy$$

and inductively

$$(2.2.14) \quad G_n(x,t) = \int_0^1 G_1(x,y) G_{n-1}(y,t) dy \quad n = 2, 3, \dots$$

From (2.2.10) and (2.2.13) it follows that

$$(2.2.15) \quad G_2(x,t) \geq 0, \quad G_3(x,t) \leq 0; \\ 0 \leq x \leq 1, \quad 0 \leq t \leq 1.$$

In general

$$(2.2.16) \quad (-1)^n G_n(x, t) \geq 0 ;$$

$$0 \leq x \leq 1, \quad 0 \leq t \leq 1 .$$

Finally, let us define

$$(2.2.17) \quad h(x) = \int_0^1 G_n(x, t) q(t) dt .$$

We note again that  $h(x)$  uniquely satisfies

$$(2.2.18) \quad \begin{aligned} h^{(2n)}(x) &= q(x) \\ h^{(2k)}(0) &= h^{(2k)}(1) = 0, \quad k = 0, 1, \dots, n-1 . \end{aligned}$$

We also need some of the known properties of Euler polynomials introduced in (2.1.7) and (2.1.8). We can easily verify that

$$(2.1.19) \quad \begin{aligned} Q_{2n}''(x) &= Q_{2n-2}(x) \\ Q_{2n}(0) &= Q_{2n}(1) = 0 . \end{aligned}$$

Furthermore,

$$(2.2.20) \quad \begin{aligned} Q_{2n}^{(2p)}(0) &= Q_{2n}^{(2p)}(1) = 0, \quad p = 0, 1, \dots, n-1, \\ Q_{2n}^{(2n)}(1) &= Q_{2n}^{(2n)}(0) = (-1)^n \\ Q_{2n}^{(2j)}(x) &= (-1)^j Q_{2n-2j}(x) \end{aligned}$$

Using (2.2.13) we note that

$$\begin{aligned} Q_2(x) &= - \int_0^1 G_1(x, t) dt , \\ Q_4(x) &= - \int_0^1 G_1(x, t) Q_2(t) dt \\ &= \int_0^1 G_1(x, t) \left[ \int_0^1 G_1(t, y) dy \right] dt \\ &= \int_0^1 G_2(x, t) dt , \end{aligned}$$

and in general,

$$(2.2.21) \quad Q_{2m}(x) = (-1)^m \int_0^1 G_m(x, t) dt .$$

Explicitly some of the first Euler polynomials are given by

$$\begin{aligned} Q_2(x) &= \frac{x(1-x)}{2!} , \quad Q_4(x) = \frac{x^2(1-x)^2 + x(1-x)}{4!} , \\ Q_6(x) &= \frac{x^3(1-x)^3 + 3x^2(1-x)^2 + 3x(1-x)}{6!} . \end{aligned}$$

### Proof of Theorem 2.1

Let  $P_{2m-1}$  denote the class of polynomials of degree  $\leq 2m-1$ . Following the notation used by Birkhoff and Priver [1967] we shall denote

$$(2.3.1) \quad G_m^{(i,j)}(x, t) = \frac{\partial^{i+j} G_m(x, t)}{\partial x^i \partial t^j} .$$

Since  $L_{2m-1}[u, x] = u(x)$  for  $u(x) \in P_{2m-1}$  it follows from the Peano theorem that for  $u \in C^{2m}[0, 1]$

$$\begin{aligned} (2.3.2) \quad e(x) &=: u(x) - L_{2m-1}[u, x] \\ &= \int_0^1 G_m(x, t) u^{(2m)}(t) dt \end{aligned}$$

where  $G_m(x, t)$  is the Peano kernel defined by (2.1.10) and (2.2.14). Differentiating (2.3.2) we have

$$\begin{aligned} (2.3.3) \quad e^{(2j)}(x) &= u^{(2j)}(x) - L_{2m-1}^{(2j)}[u, x] \\ &= \int_0^1 G_m^{(2j, 0)}(x, t) u^{(2m)}(t) dt . \end{aligned}$$

Let us substitute  $u(x) = Q_{2m}(x)$  (as defined by (2.1.7)) in (2.3.3) and use various properties as given by

(2.2.20) and (2.2.21). We then obtain

$$(2.3.4) \quad Q_{2m}^{(2j)}(x) - L_{2m-1}^{(2j)}[Q_{2m}, x] \\ = \int_0^1 G_m^{(2j,0)}(x,t) Q_{2m}^{(2m)}(t) dt.$$

We know from (2.2.20)

$$(2.3.5) \quad Q_{2m}^{(2m)}(t) = Q_{2m}^{(2p)}(0) = (-1)^m.$$

Moreover,

$$(2.3.6) \quad Q_{2m}^{(2p)}(0) = Q_{2m}^{(2p)}(1) = 0, \quad p = 0, 1, \dots, m-1.$$

It follows that

$$(2.3.7) \quad L_{2m-1}[Q_{2m}, x] = 0$$

identically. Thus (2.3.4) can be rewritten as

$$(2.3.8) \quad Q_{2m}^{(2j)}(x) = (-1)^m \int_0^1 G_m^{(2j,0)}(x,t) dt.$$

Next we note from (2.2.14) that

$$G_m^{(2,0)} = G_{m-1}(x,t).$$

Hence

$$G_m^{(4,0)}(x,t) = G_{m-1}^{(2,0)}(x,t) = G_{m-2}(x,t)$$

and in general,

$$(2.3.9) \quad G_m^{(2j,0)}(x,t) = G_{m-j}(x,t).$$

From (2.2.16) and (2.3.9) we have

$$(2.3.10) \quad (-1)^{m-j} G_m^{(2j,0)}(x,t) = (-1)^{m-j} G_{m-j}(x,t) \geq 0$$

in the unit square  $0 \leq x \leq 1$ ,  $0 \leq t \leq 1$ .

Combining (2.3.3), (2.3.9), (2.3.10), (2.2.19), and (2.3.8), it follows that

$$|e^{(2j)}(x)| \leq \int_0^1 |G_m^{(2j,0)}(x,t)| dt$$

$$\begin{aligned}
&= U \left| \int_0^1 G_m^{(2j,0)}(x,t) dt \right| \\
&= U Q_{2m-2j}(x) , \\
&\quad j = 0, 1, \dots, m-1 .
\end{aligned}$$

This proves (2.1.10).

We next turn to prove (2.1.11). Due to (2.3.9), it is enough to prove (2.1.11) for  $j = 1$ . From (2.2.14), it follows that

$$\begin{aligned}
(2.3.11) \quad G_m^{(1,0)}(x,t) &= \int_0^x y G_{m-1}(y,t) dy \\
&\quad + \int_x^1 (y-1) G_{m-1}(y,t) dy .
\end{aligned}$$

Therefore

$$\begin{aligned}
(2.3.12) \quad \int_0^1 |G_m^{(1,0)}(x,t)| dt \\
\leq \int_0^1 \int_0^x y |G_{m-1}(y,t)| dy dt \\
+ \int_0^1 \int_x^1 (1-y) |G_{m-1}(y,t)| dy dt .
\end{aligned}$$

Recalling (2.2.21)

$$Q_{2m-2}(y) = (-1)^{m-1} \int_0^1 G_{m-1}(y,t) dt ,$$

$$m = 2, 3, \dots$$

and the fact that in the unit square  $0 \leq x \leq 1$ ,  $0 \leq t \leq 1$ ,

$$(-1)^{m-1} G_{m-1}(y,t) \geq 0 ,$$

we can assert that

$$(2.3.13) \quad Q_{2m-2}(y) = \int_0^1 |G_{m-1}(y,t)| dt.$$

On changing the order of integration in (2.3.12) and making use of (2.3.13), we obtain

$$(2.3.14) \quad \int_0^1 |G_m^{(1,0)}(x,t)| dt \leq \int_0^x y Q_{2m-2}(y) dy \\ + \int_x^1 (1-y) Q_{2m-2}(y) dy \\ =: \chi_{2m-2}(x).$$

Using (2.2.20) we note that

$$(2.3.15) \quad \chi_{2m-2}(x) = - \int_0^x y Q_{2m}'(y) dy \\ - \int_x^1 (1-y) Q_{2m}'(y) dy.$$

On integrating by parts, we have

$$(2.3.16) \quad \chi_{2m-2}(x) = - y Q_{2m}'(y) \Big|_0^x + \int_0^x Q_{2m}'(y) dy \\ - Q_{2m}'(y) (1-y) \Big|_x^1 + \int_x^1 -Q_{2m}'(y) dy \\ = -x Q_{2m}'(x) + (1-x) Q_{2m}'(x) + 2Q_{2m}(x) \\ = (1-2x) Q_{2m}'(x) + 2Q_{2m}(x).$$

Also

$$(2.3.17) \quad \chi_{2m-2}'(x) = (1-2x) Q_{2m}''(x).$$

Since  $Q_{2m-2}$  vanishes only at  $x = 0$  and  $x = 1$ , it follows that the critical point at  $\chi_{2m-2}(x)$  inside  $[0,1]$  is only at  $x = 1/2$ . Also we note that  $\chi_{2m-2}(1) = \chi_{2m-2}(0)$ .

Further

$$\begin{aligned}
 (2.3.18) \quad \chi_{2m-2}(1) - \chi_{2m-2}(1/2) \\
 = \int_{1/2}^1 (2x-1) Q_{2m-2}(y) dy > 0.
 \end{aligned}$$

Thus we conclude that  $\chi_{2m-2}(x)$  has an absolute maximum at  $x = 0$  and  $x = 1$ . Therefore, from (2.3.2), (2.3.14), and (2.3.11), it follows that

$$\begin{aligned}
 (2.3.19) \quad |e'(x)| &\leq U \int_0^1 |G_m^{(1,0)}(x,t)| dt \\
 &\leq U \chi_{2m-2}(x) \\
 &\leq U |(1-2x) Q_{2m}'(x) + 2Q_{2m}(x)| \\
 &\leq U \chi_{2m-2}(1).
 \end{aligned}$$

On using (2.3.15) it follows that

$$|e'(x)| \leq U \chi_{2m-2}(1) = -U Q_{2m}'(1) = U Q_{2m}'(0),$$

which proves (2.1.12).

That (2.1.11) and (2.1.12) are best possible follows from the Peano theorem, or more simply, by choosing  $u(x) = Q_{2m}(x)$ , the Euler polynomial defined by (2.1.7). In view of (2.2.20), we have  $U =: \max_{0 \leq x \leq 1} |u^{(2m)}(x)| = 1$ . Further use of (2.2.20) and the definition of  $L_{2m-1}[u, x]$  show that  $L_{2m-1}[Q_{2m}, x]$  is identically zero. Our choice of  $u(x)$  then gives pointwise equality in (2.1.11). Similarly it can be shown that (2.1.12) is also pointwise best possible. This proves the theorem.

It is perhaps worth remarking that any exact evaluation of the integral of the absolute value of a Peano kernel results in an exact error bound (see Sard

[1963] or Stroud [1974])). Generally error bounds resulting from integration of a Peano kernel under the assumption that  $u(x) \in C^k[a,b]$  also hold for  $u$  having piecewise continuous  $k$ th derivative on  $[a,b]$ , and even for  $u$  having  $(k-1)$ st derivative absolutely continuous on  $[a,b]$ . In the case given here we can thus expand the class of functions for which the error bounds of Theorem 2.1 hold and hence are best possible.

As Theorem 2.1 is stated for function  $u(x)$   $2m$  times continuously differentiable, it also holds when the  $2m$ th derivative is merely piecewise continuous on  $[0,1]$ . Moreover the theorem holds even for the case that  $u(x)$  has its  $(2m-1)$ st derivative absolutely continuous. In this last case  $U$ , instead of being the max of the  $2m$ th derivative on  $[0,1]$ , becomes the "L infinity" norm of the generalized  $2m$ th derivative. In the following chapters the classes of functions  $k$  times continuously differentiable, the class of functions having piecewise continuous  $k$ th derivative and the class having  $k-1$ st derivative absolutely continuous may be treated as being interchangeable.

CHAPTER THREE  
MORE POLYNOMIAL ERROR BOUNDS

Introduction and Statement of Theorems

Let  $u \in C^{(2m+2)}[0, h]$  be given. It follows from a result of Schoenberg [1966] that there exists a unique polynomial  $w_{2m+1}[u, x]$  of degree  $\leq 2m+1$  satisfying

$$(3.1.1) \quad \begin{aligned} w_{2m+1}[u, 0] &= u(0) , & w_{2m+1}[u, h] &= u(h) , \\ w_{2m+1}^{(p)}[u, 0] &= u^{(p)}(0) , \\ w_{2m+1}^{(p)}[u, h] &= u^{(p)}[u, h] , \end{aligned}$$

$$p = 2, 3, \dots, m+1 .$$

Theorems 3.1 and 3.2 will give bounds on  $u^{(j)}(x) - w_{2m+1}^{(j)}(x)$  for the cases  $m = 2$  and  $m = 3$  of polynomials  $w_{2m+1}$  satisfying (3.1.1).

The polynomial  $w_{2m+1}[u, x]$  can be expressed in relation to the Hermite polynomial  $v_{2m-1}[u'', x]$ . To illustrate the relation between  $w_{2m+1}$  and  $v_{2m-1}$ , let  $h = 1$  and let  $v_{2m-1}[g, x]$  be the Hermite polynomial of degree at most  $2m - 1$  matching  $g =: u''$  and its first  $m - 1$  derivatives at 0 and 1. We can represent  $v_{2m-1}[g, x]$  as

$$(3.1.2) \quad \begin{aligned} v_{2m-1}[g, x] &= A_0(x)g(0) + B_0(x)g(1) \\ &+ A_1(x)g'(0) + B_1(x)g'(1) \\ &+ A_2(x)g''(0) + B_2(x)g''(1) \\ &\quad \cdot \\ &+ A_{m-1}(x)g^{(m-1)}(0) + B_{m-1}(x)g^{(m-1)}(1) \end{aligned}$$

where  $A_i(x)$  and  $B_i(x)$ ,  $i = 0, 1, \dots, m-1$  are polynomials of degree  $2m-1$  or less satisfying

$$(3.1.3) \quad A_i^{(j)}(0) = \delta_{ij}, \quad A_i^{(j)}(1) = 0,$$

$$j = 0, 1, \dots, m-1$$

$$B_i^{(j)}(0) = 0, \quad B_i^{(j)}(1) = \delta_{ij},$$

$$j = 0, 1, \dots, m-1.$$

Define for  $i = 0, 1, \dots, m-1$

$$(3.1.4) \quad C_i(x) = \int_0^1 G_1(x,t) A_i(t) dt,$$

$$D_i(x) = \int_0^1 G_1(x,t) B_i(t) dt,$$

From (3.1.4), (3.1.3) and (2.2.5)-(2.2.8), it follows that for  $i = 0, 1, \dots, m-1$

$$(3.1.5) \quad C_i^{(j)}(0) = \delta_{i(j-2)}, \quad C_i^{(j)}(1) = 0,$$

$$D_i^{(j)}(0) = 0, \quad D_i^{(j)}(1) = \delta_{i(j-2)};$$

$$j = 2, 3, \dots, m+1$$

where

$$(3.1.6) \quad C_i(0) = D_i(0) = C_i(1) = D_i(1) = 0,$$

and  $C_i, D_i$  are polynomials of degree  $2m-1$  or less.

For a given  $u \in C^{(2m)}[0,1]$  we can use (3.1.5) and (3.1.6) to give  $w_{2m+1}[u,x]$  in the form

$$(3.1.7) \quad w_{2m+1}[u,x] = u(0)(1-x) + u(1)x \\ + u''(0)C_0(x) + u''(1)D_0(x) \\ + u^{(3)}(0)C_1(x) + u^{(3)}(1)D_1(x) \\ + \quad \cdot \quad \cdot \quad \cdot \\ + u^{(m+1)}(0)C_{m-1}(x) + u^{(m+1)}(1)D_{m-1}(x).$$

For  $m = 2$  and  $m = 3$ , we give (3.1.7) explicitly. For  $m = 2$ , if  $u \in C^{(6)}[0,1]$ , then the unique quintic  $w_5[u,x]$  matching  $u$  and its second and third derivatives at 0 and 1 is given by

$$\begin{aligned}
 (3.1.8) \quad w_5[u,x] = & (1-x) u(0) + x u(1) \\
 & + u''(0) \left[ -7x/20 + x^2/2 - x^4/4 + x^5/10 \right] \\
 & + u''(1) \left[ -3x/20 + x^4/4 - x^5/10 \right] \\
 & + u'''(0) \left[ -x/20 + x^3/6 - x^4/6 + x^5/20 \right] \\
 & + u'''(1) \left[ x/30 - x^4/12 + x^5/20 \right] .
 \end{aligned}$$

For  $u \in C^{(8)}[0,1]$ , the unique polynomial  $w_7[u,x]$  of degree  $\leq 7$ , matching  $u$  and its second, third and fourth derivatives at 0 and 1 is given by

$$\begin{aligned}
 (3.1.9) \quad w_7[u,x] = & (1-x) u(0) + x u(1) \\
 & + u''(0) \left[ -5x/14 + x^2/2 - x^5/2 + x^6/2 - x^7/7 \right] \\
 & + u''(1) \left[ -x/7 + x^5/2 - x^6/2 + x^7/7 \right] \\
 & + u^{(3)}(0) \left[ -13x/210 + x^3/6 - 3x^5/10 \right. \\
 & \quad \left. + 4x^6/15 - x^7/14 \right] \\
 & + u^{(3)}(1) \left[ 4x/105 - x^5/5 + 7x^6/30 - x^7/14 \right] \\
 & + u^{(4)}(0) \left[ -x/210 + x^4/24 - 3x^5/40 \right. \\
 & \quad \left. + x^6/20 - x^7/84 \right] \\
 & + u^{(4)}(1) \left[ -x/280 + x^5/40 - x^6/30 + x^7/84 \right] .
 \end{aligned}$$

The following theorem concerns the quintic interpolant  $w_5$ .

**Theorem 3.1** Let  $u \in C^6[0,1]$  and let  $w_5[u,x]$  satisfy

$$\begin{aligned}
 (3.1.10) \quad w_5^{(p)}[u,0] &= u^{(p)}(0) , \\
 w_5^{(p)}[u,1] &= u^{(p)}(1) , \quad p = 0, 2, 3 .
 \end{aligned}$$

Denote

$$(3.1.11) \quad e(x) = u(x) - w_5[u, x]$$

and

$$(3.1.12) \quad U = \max_{0 \leq x \leq 1} |u^{(6)}(x)|.$$

Then for  $0 \leq x \leq 1$ ,  $p = 0, 1, 2, 3, 4, 5$ , the following pointwise bounds hold:

$$(3.1.13) \quad |e^{(p)}(x)| \leq U f_{0,p}(x)$$

where

$$f_{0,0}(x) = [x^3(1-x)^3 + x^2(1-x)^2/2 + x(1-x)/2] / 6!,$$

$$f_{0,1}(x) = [1/60 - x^3(1-x)^3/3] / 4!,$$

$$f_{0,2}(x) = [x^2(1-x)^2] / 4!,$$

$$f_{0,3}(x) = [x(x-1)(2x-1)] / 12, \quad 0 \leq x \leq 1/3$$

$$= [16x^3 - 105x^2 + 197x - 162$$

$$+ 66/x - 13/x^2 + 1/x^3] / 96,$$

$$1/3 \leq x \leq 1/2$$

$$f_{0,4}(x) = [48x^5 + 42x^4 - 100x^3$$

$$+ 54x^2 - 12x + 1] / 12(1-x)^3,$$

$$0 \leq x \leq 1/3$$

$$= [-6(x-1/2)^2 + 1/2] / 12, \quad 1/3 \leq x \leq 2/3$$

$$f_{0,5}(x) = -(x-1/2)^4 + 3(x-1/2)^2/2 + 3/16, \quad 0 \leq x \leq 1$$

and where  $f_{0,2}$  and  $f_{0,3}$  are extended to the whole of  $[0,1]$  by even symmetry about  $1/2$ .

Furthermore, the functions  $f_{0,p}$ ,  $p = 0, 2, 3, 4$ , and  $5$  are pointwise best possible. The functions  $f_{0,2}$ ,  $f_{0,3}$ ,  $f_{0,4}$  and  $f_{0,5}$  are those of Birkhoff and Priver [1967] for two point cubic interpolation.

That these functions also serve as error bounds in

the present case is a consequence of the fact that  $w_5''[u, x]$  is the unique cubic matching  $u''$  and  $u'''$  at 0 and 1. In other words  $w_5''[u, x]$  is the Hermite cubic interpolation  $v_3[g, x]$  where  $g = u''$ . The error bounds given by Birkhoff and Priver in terms of  $\max_{0 \leq x \leq 1} |g^{(4)}(x)|$  are now expressed in terms of  $U = \max_{0 \leq x \leq 1} |u^{(6)}(x)|$  (as  $g^{(4)}$  is in fact  $u^{(6)}$ ).

Denoting

$$(3.1.14) \quad c_p = \max_{0 \leq x \leq 1} |f_{0,p}(x)| \quad p = 0, 1, \dots, 5$$

we have

$$c_0 = \frac{11}{6^4} \frac{1}{6!}, \quad c_1 = \frac{1}{2} \frac{1}{6!}, \quad c_2 = \frac{1}{2^4} \frac{1}{4!}$$

$$c_3 = \frac{1\sqrt{3}}{9} \frac{1}{4!}, \quad c_4 = \frac{1}{12}, \quad c_5 = \frac{1}{2}.$$

From (3.1.14) and (3.1.13) it follows that for every  $u \in C^{(6)}[0, 1]$

$$(3.1.15) \quad \max_{0 \leq x \leq 1} |e^{(p)}(x)| \leq c_p U; \quad p = 0, 1, \dots, 5.$$

Remark 3.1 Note that

$$c_p = \max_{0 \leq x \leq 1} |f_{0,0}^{(p)}(x)| \quad p = 0, 1, \dots, 5.$$

If we set  $u(x) = f_{0,0}(x)$  then we have

$$\begin{aligned} e(x) &= f_{0,0}(x) - w_7[f_{0,0}, x] \\ &= f_{0,0}(x) \text{ and } U = \max_{0 \leq x \leq 1} |f_{0,0}^{(6)}(x)| = 1. \end{aligned}$$

By Remark 3.1 we see that for  $u(x) = f_{0,0}(x)$  equality is attained in (3.1.15) for  $p = 0, 1, 2, 3, 4, 5$ . The constants  $c_p$  are thus the smallest possible.

The next theorem gives error bounds for  $w_7$ , analogous to the error bounds for  $w_5$  given in Theorem 3.1.

Theorem 3.2 Let  $u \in C^{(8)}[0,1]$ , and let  $w_7[u,x]$  be a polynomial of degree 7 or less satisfying

$$(3.1.16) \quad w_7^{(p)}[u,0] = u^{(p)}(0) , \\ w_7^{(p)}[u,1] = u^{(p)}(1) , \quad p = 0, 2, 3, 4 .$$

Denote

$$(3.1.17) \quad e(x) = u(x) - w_7[u,x]$$

and denote

$$(3.1.18) \quad U = \max_{0 \leq x \leq 1} |u^{(8)}(x)| .$$

Then, for  $0 \leq x \leq 1$  and  $0 \leq p \leq 7$ , the following pointwise bounds hold:

$$(3.1.19) \quad |e^{(p)}(x)| \leq U f_{1,p}(x)$$

where

$$\begin{aligned} f_{1,0}(x) &= [ x^4(1-x)^4 + (2/5)x^3(1-x)^3 \\ &\quad + x^2(1-x)^2/5 + x(1-x)/5 ] / 8! , \\ f_{1,1}(x) &= (1/5)(1/8!) - (1/4)(1/6!)x^4(1-x)^4 , \\ f_{1,2}(x) &= x^3(1-x)^3/6! , \\ f_{1,3}(x) &= x^2(x-1)^2(1-2x)/240 , \quad 0 \leq x \leq 2/5 \\ &= x^2(x-1)^2(1-2x)/240 \\ &\quad + T^4(x-1)^2 [ 10T^2x^2 \\ &\quad + 2T(-15x^2+2x+1) + 5x(5x-2) ] / 120 , \\ &\quad 2/5 \leq x \leq 1/2 \end{aligned}$$

where

$$\begin{aligned} T &= [ (3x-1)(5x+1) + (x-1)(-15x^2+6x+1)^{1/2} ] / 12 \\ f_{1,4}(x) &= x(1-x)(5x^2-5x+1)/120 , \quad 0 \leq x \leq (4-\sqrt{6})/10 \\ &= x(1-x)(5x^2-5x+1)/120 \\ &\quad + T_1^4 [ 2T_1^2(2x^3-3x^2+x) \end{aligned}$$

$$\begin{aligned}
& + 12T_1(-5x^3+8x^2-3x)/5 \\
& + (10x^3-18x^2+9x-1) ] / 12 , \\
& \text{for } (4-\sqrt{6})/10 \leq x \leq (3-\sqrt{3})/6
\end{aligned}$$

where

$$\begin{aligned}
T_1 &= [ 15x^2 - 9x - (x-1)(3x(4-5x))^{1/2} ] / 6x(2x-1) , \\
f_{1,4}(x) &= x(x-1)(5x^2-5x+1)/120 \\
&+ W^4x [ 10W^2(2x^2-3x+1) \\
&+ 4W(15x^2-21x+6) \\
&+ 5(10x^2-12x+3) \\
&+ 5(10x^2-12x+3) ] / 60 , \\
&\text{for } (3-\sqrt{3})/6 \leq x \leq (6-\sqrt{6})/10
\end{aligned}$$

and where

$$\begin{aligned}
W &= [ \frac{3(1-x)(5x-2) + x(3(1-x)(5x-1))^{1/2}}{6(x-1)(2x-1)} ] , \\
f_{1,4}(x) &= x(x-1)(5x^2-5x+1)/120 , \\
&(6-\sqrt{6})/10 \leq x \leq 1/2 \\
f_{1,5}(x) &= (2x-1)(10x^2-10x+1)/120 \\
&+ W_1^4 [ 20W_1^2(6x^2-6x+1) \\
&+ 24W_1(15x^2-14x+2) \\
&+ 30(10x^2-8x+1) ] / 120 , \\
&0 \leq x \leq (4-\sqrt{6})/10
\end{aligned}$$

where

$$\begin{aligned}
W_1 &= [ \frac{-15x^2 + 14x - 2 - x(3x(4-5x))^{1/2}}{12x^2 - 12x + 2} ] \\
&= (2x-1)(10x^2-10x+1)/120 , \\
&(4-\sqrt{6})/10 \leq x \leq (6-\sqrt{6})/10
\end{aligned}$$

$$\begin{aligned}
&= (2x-1)(10x^2-10x+1)/120 \\
&\quad - T_2^4 [ 20T_2^2(6x^2-6x+1) \\
&\quad + 24T_2(-15x^2+16x-3) \\
&\quad + 30(10x^2-12x+3) ] / 120 , \\
&\quad (6-\sqrt{6})/10 \leq x \leq 1/2
\end{aligned}$$

where

$$\begin{aligned}
T_2 &= [ \frac{15x^2 - 16x + 3 - (x-1)(-15x^2+18x-3)^{1/2}}{12x^2 - 12x + 2} ] \\
f_{1,6}(x) &= [ -15x^2 + 5x - 1 ] / 10 \\
&\quad - W_2^4 [ W_2^2(x-1/2) \\
&\quad + W_2(15x-7)/5 + 5x/2 - 1 ] , \\
&\quad 0 \leq x \leq 2/5
\end{aligned}$$

where

$$\begin{aligned}
W_2 &= [ -15x + 7 - (-15x+6x+1)^{1/2} ] / (12x-6) \\
f_{1,6}(x) &= -(x-1/2)^{2/2} + 1/40 , \quad 2/5 \leq x \leq 1/2 \\
f_{1,7}(x) &= 2(x-1/2)^6 - 5(x-1/2)^4/2 \\
&\quad + 15(x-1/2)^2/8 + 5/32 , \\
&\quad 0 \leq x \leq 1
\end{aligned}$$

and where  $f_{1,3}$ ,  $f_{1,4}$ ,  $f_{1,5}$ , and  $f_{1,6}$  are extended to  $[1/2, 1]$  by symmetry about  $x = 1/2$ . Furthermore, each of the functions  $f_{1,p}$  where  $p = 0, 2, 3, \dots, 7$  is pointwise exact.

Setting  $d_p = \max_{0 \leq x \leq 1} |f_{1,p}(x)|$ , we have

$$(3.1.20) \quad d_0 = \left(\frac{93}{1280}\right) \frac{1}{8!}, \quad d_1 = \left(\frac{1}{5}\right) \frac{1}{8!}$$

$$d_2 = \frac{1}{6!} \frac{1}{2^6}, \quad d_3 = \frac{5}{30,000}$$

$$d_4 = \frac{1}{1920} ,$$

$$d_5 = \frac{1}{120} ,$$

$$d_6 = \frac{1}{10} ,$$

$$d_7 = \frac{1}{2} .$$

From (3.1.19) and (3.1.20) it follows that for

$$0 \leq p \leq 7$$

$$(3.1.21) \quad \max_{0 \leq x \leq 1} |e^{(p)}(x)| \leq d_p U .$$

Remark 3.2 Analogously to Remark 3.1, note that

$$(3.1.22) \quad \max_{0 \leq x \leq 1} |f_{1,p}(x)| = \max_{0 \leq x \leq 1} |f_{1,0}^{(p)}(x)| .$$

On setting  $u = f_{1,0}(x)$  it follows from (3.1.22) that

(3.1.21) is exact for each  $p$ .

The following would seem to a natural generalization of the Theorems 3.1 and 3.2.

Conjecture 3.3 Let  $u \in C^{(2m+2)}[0,1]$  and let  $w_{2m+1}[u,x]$  be the polynomial of degree at most  $2m+1$  matching  $u$  and its 2nd, 3rd, . . . ,  $(m+1)$ st derivatives at 0 and 1.

Denote

$$(3.1.23) \quad e(x) = u(x) - w_{2m+1}[u,x]$$

and

$$(3.1.24) \quad U = \max_{0 \leq x \leq 1} |u^{(2m+2)}(x)| .$$

Then for  $p = 0, 1, 2$ , we have

$$(3.1.25) \quad |e^{(p)}(x)| \leq U f_{m-1,p}(x)$$

where

$$f_{m-1,0}(x) = \left\{ \sum_{i=0}^m (-1)^i \binom{m}{i} \frac{[x^{m+2+i} - x]}{[(m+i+2)(m+i+1)]} \right\} / (2m)! ,$$

$$f_{m-1,1}(x) = \left\{ \frac{1}{(2m+2) \binom{2m+1}{m}} - \frac{x^{m+1}(1-x)^{m+1}}{m+1} \right\} / (2m)! ,$$

$$f_{m-1,2}(x) = \{ x^m(1-x)^m \} / (2m)! .$$

Furthermore (3.1.25) is pointwise exact,  $p = 0$  and  $2$ .

Analogously to Remarks 3.1 and 3.2, it may be that for every  $u \in C^{(2m+2)}[0,1]$  and  $p = 0, 1, \dots, 2m+1$

$$(3.1.26) \quad \max_{0 \leq x \leq 1} |e^{(p)}(x)| \leq U \max_{0 \leq x \leq 1} |f_{m-1,0}^{(p)}(x)| .$$

If Equation (3.1.26) holds then it is best possible as can be verified by choosing  $u = f_{m-1,0}$  and noting that then  $e(x)$  is the same as  $f_{m-1,0}(x)$ . For  $p = 0, 1, 2$ ,

$$\max_{0 \leq x \leq 1} |f_{m-1,p}(x)| = \max_{0 \leq x \leq 1} |f_{m-1,0}^{(p)}(x)| .$$

Hence if (3.1.25) holds then (3.1.26) is true for

$p = 0, 1, 2$ . As

$$f_{m-1,2}^{(2)}(x) = [x^m(1-x)^m] / (2m)! ,$$

the conjecture of (3.1.26) is related to the following conjecture.

Conjecture 3.4 Let  $u \in C^{(2m)}[0,1]$  and let  $v_{2m-1}$  be the Hermite polynomial of degree at most  $2m-1$  matching  $u$  and its first  $m-1$  derivatives at  $0$  and  $1$ . Denote

$$U = \max_{0 \leq x \leq 1} |u^{(2m)}(x)|$$

and

$$e(x) = v_{2m-1}[u, x] - u(x) .$$

Then

$$\begin{aligned} \max_{0 \leq x \leq 1} |e^{(p)}(x)| \\ \leq U \max_{0 \leq x \leq 1} \left| \frac{d^p}{dx^p} [x^m(1-x)^m / (2m)!] \right| , \\ p = 0, 1, 2, \dots, 2m-1 . \end{aligned}$$

The results of Birkhoff and Priver demonstrate Conjecture 3.4 for the cases  $m = 2$  and  $m = 3$ . Recent work of Bojanov and Varma indicates that Conjecture 3.4 is in fact true.

The next theorem will concern an interpolatory polynomial which enjoys a similar property to that of the above conjectures. Let  $u \in C^{(4)}[0,1]$ . Define  $k_3[u,x]$  by

$$(3.1.27) \quad k_3[u,x] = u(0) (1-x) (1-2x)^2 + u(1/2) 4x(1-x) \\ + u(1) x(1-2x)^2 + u'(1/2) 2x(1-x) (2x-1) .$$

Then  $k_3[u,x]$  is the unique polynomial of degree 3 or less satisfying

$$(3.1.28) \quad k_3[u,x] = u(0) , \quad k_3[u,1] = u(1) , \\ k_3[u,1/2] = u(1/2) , \quad k_3'[u,1/2] = u'(1/2) .$$

Theorem 3.3. Let  $u \in C^{(4)}[0,1]$ . Denote

$$e(x) = k_3[u,x] - u(x) ,$$

$$U = \max_{0 \leq x \leq 1} |u^{(4)}(x)| .$$

Then for  $p = 0, 1, 2, 3$ , we have

$$(3.1.29) \quad |e^{(p)}(x)| \leq a_p U$$

where

$$a_0 = 1 / (2^8 4!) , \quad a_1 = 1 / (2^2 4!) ,$$

$$a_2 = (5/2) (1/4!) , \quad a_3 = 1/2 .$$

That the  $a_p$  are the best possible can be verified by choosing

$$u(x) = [ x(1-x) (1-2x)^2 ] / (2^2 4!) .$$

Due to the similarity between the proof of Theorem 3.3 and several other proofs in the following chapters, it would be redundant to prove it here.

Proof of Theorem 3.1

Let  $u \in C^{(6)}[0,1]$ . Then

$$\begin{aligned}
 (3.2.1) \quad w_5[u,x] &= (1-x) u(0) + x u(1) \\
 &\quad + u''(0) \left[ -7x/20 + x^2/2 - x^4/4 + x^5/10 \right] \\
 &\quad + u''(1) \left[ -3x/20 + x^4/4 - x^5/10 \right] \\
 &\quad + u'''(0) \left[ -x/20 + x^3/6 - x^4/6 + x^5/20 \right] \\
 &\quad + u'''(1) \left[ x/30 - x^4/12 + x^5/20 \right]
 \end{aligned}$$

is the only polynomial of degree  $\leq 5$  satisfying

$$\begin{aligned}
 (3.2.2) \quad w_5^{(p)}[u,0] &= u^{(p)}(0) , \\
 w_5^{(p)}[u,1] &= u^{(p)}(1) , \quad p = 0, 2, 3 .
 \end{aligned}$$

Define

$$(3.2.3) \quad e(x) = u(x) - w_5[u,x] .$$

Then

$$(3.2.4) \quad e^{(p)}(0) = 0, e^{(p)}(1) = 0, p = 0, 2, 3,$$

and

$$(3.2.5) \quad e^{(6)}(x) = Q(x) =: u^{(6)}(x) .$$

In other words,  $e(x)$  is the unique solution of the differential equation (3.2.5) with boundary conditions (3.2.4). We can rephrase (3.2.4) and (3.2.5) as

$$\begin{aligned}
 (3.2.6) \quad \frac{d^2 e}{dx^2} &= y(x) , \\
 e(0) &= 0, e(1) = 0 ,
 \end{aligned}$$

and

$$\begin{aligned}
 (3.2.7) \quad \frac{d^4 y}{dx^4} &= Q(x) , \\
 y(0) &= y(1) = y'(0) = y'(1) = 0 .
 \end{aligned}$$

From (3.2.6) and (3.2.2)-(3.2.6), it follows that

$$(3.2.8) \quad e(x) = \int_0^1 G_1(x, z) y(z) dz$$

where

$$G_1(x, z) = \begin{cases} z(x-1), & 0 \leq z < x \leq 1 \\ x(z-1), & 0 \leq x \leq z \leq 1 \end{cases}$$

is the Peano kernel for linear interpolation used in the proof of Theorem 2.1.

Similarly, from Birkhoff and Priver (or by application of the Peano theorem), we have

$$(3.2.9) \quad y(z) = \int_0^1 G_4(z, t) Q(t) dt,$$

where

$$6G_4(z, t) = \begin{cases} (3t^2 - 2t^3)z^3 + 3(t-2)t^2z^2 + 3t^2z - t^3, & t \leq z \\ (3t^2 - 2t^3 - 1)z^3 + 3(t-1)^2tz^2, & t > z \end{cases}$$

for  $0 \leq t \leq 1, \quad 0 \leq z \leq 1$ .

Combining (3.2.8) and (3.2.9), we have

$$\begin{aligned} (3.2.10) \quad e(x) &= \int_0^1 G_1(x, z) \int_0^1 G_4(z, t) Q(t) dt dz \\ &= \int_0^1 G_1(x, z) \int_0^1 G_4(z, t) u^{(6)}(t) dt dz \\ &= \int_0^1 \int_0^1 G_1(x, z) G_4(z, t) u^{(6)}(t) dt dz \\ &= \int_0^1 \int_0^1 G_1(x, z) G_4(z, t) dz u^{(6)}(t) dt \end{aligned}$$

$$= \int_0^1 G(x,t) u^{(6)}(t) dt$$

where

$$(3.2.11) \quad G(x,t) = \int_0^1 G_1(x,z) G_4(z,t) dz .$$

From (3.2.11) and (2.2.5)-(2.2.8), it follows that

$$(3.2.12) \quad G^{(2,0)}(x,t) = G_4(x,t)$$

and

$$(3.2.13) \quad G^{(p+2,0)}(x,t) = G_4^{(p,0)}(x,t) , \quad p = 0, 1, 2, 3 .$$

Also, as

$$G_4(z,t) \leq 0 , \quad 0 \leq z \leq 1 , \quad 0 \leq t \leq 1$$

$$G_1(x,z) \leq 0 , \quad 0 \leq x \leq 1 , \quad 0 \leq z \leq 1$$

it follows that

$$G(x,t) \geq 0 \quad 0 \leq x \leq 1 , \quad 0 \leq t \leq 1 .$$

From (3.2.10) and  $G(x,t) \geq 0$ , we have

$$(3.2.14) \quad |e(x)| \leq \int_0^1 G(x,t) dt \max_{0 \leq x \leq 1} |u^{(6)}(x)| .$$

In fact,

$$(3.2.15) \quad \int_0^1 G(x,t) dt = \iiint_0^1 G_4(x,t) dt dx dx + ax + b$$

where  $a$  and  $b$  are chosen to satisfy

$$(3.2.16) \quad \int_0^1 G(0,t) dt = \int_0^1 G(1,t) dt = 0 .$$

We know from Birkhoff and Priver (or Hermite) that

$$\int_0^1 G_4(x,t) dt = [-x^2(1-x)^2] / (4!) .$$

Then

$$\int_0^1 G_4(x,t) dt \, dx \, dx = \frac{1}{6!} \left( -\frac{5x^4}{2} + 3x^5 - x^6 \right),$$

and to satisfy (3.2.16), we have a and b of (3.2.15) as

$$a = \frac{1/2}{6!}, \quad b = 0.$$

Rearranging, we have

$$\begin{aligned} (3.2.17) \quad \int_0^1 G(x,t) dt &= \left[ -5x^4/2 + 3x^5 - x^6 + x/2 \right] / 6! \\ &= \left[ x^3(1-x)^3 + (1/2)x^2(1-x)^2 + (1/2)x(1-x) \right] / 6! \\ &= f_{0,0}(x). \end{aligned}$$

Combining (3.2.14) and (3.2.17), we have the result of the theorem for  $p = 0$ .

From 3.2.10, we have

$$(3.2.18) \quad |e^{(p)}(x)| \leq \int_0^1 |G^{(p)}(x,t)| dt \max_{0 \leq x \leq 1} |f^{(6)}(x)|.$$

From 3.2.11, we have

$$(3.2.19) \quad G^{(1,0)}(x,t) = \int_0^x y G_4(y,t) dy + \int_x^1 (y-1) G_4(y,t) dy.$$

Therefore as  $G_4(y,t) \leq 0$   $0 \leq y \leq 1$ ,  $0 \leq t \leq 1$

$$\begin{aligned} (3.2.20) \quad |G^{(1,0)}(x,t)| &\leq \int_0^x y |G_4(y,t)| dy \\ &\quad + \int_x^1 (1-y) |G_4(y,t)| dy. \end{aligned}$$

As before, we have

$$\int_0^1 |G_4(y,t)| dt = y^2(1-y)^2 / 4!.$$

Thus

$$\begin{aligned}
 (3.2.21) \quad \int_0^1 |G^{(1,0)}(x,t)| dt &\leq \int_0^1 \int_0^x |G_4(y,t)| dy dt \\
 &\quad + \int_0^1 \int_x^1 (1-y) |G_4(y,t)| dy dt \\
 &= \int_0^x y \int_0^1 |G_4(y,t)| dt dy \\
 &\quad + \int_x^1 (1-y) \int_0^1 |G_4(y,t)| dt dy \\
 &= \int_0^x [y^3(1-y)^2] / 4! dy \\
 &\quad + \int_x^1 [(1-y)^3 y^2] / 4! dy \\
 &= [1/60 - (x^3(1-x)^3)/3] / 4! \\
 &= f_{0,1}(x)
 \end{aligned}$$

which achieves its maximum value of  $1/1440$  for  $x = 0$  or  $x = 1$ . We note also that

$$\begin{aligned}
 1/1440 &= 1/(2 \cdot 6!) = C_1 \\
 &= \max_{0 \leq x \leq 1} |f_{0,0}^{(1)}(x)| = \max_{0 \leq x \leq 1} |f_{0,1}(x)|.
 \end{aligned}$$

Combining (3.18) and (3.13), we have

$$(3.2.22) \quad |e^{(p)}(x)| \leq \int_0^1 |G_4^{(p-2,0)}(x-t)| dt \max_{0 \leq x \leq 1} |u^{(6)}(x)|,$$

for  $p = 2, 3, 4, 5$ .

As this inequality is precisely that used by Birkhoff and Priver to derive the functions  $f_{0,2}$ ,  $f_{0,3}$ ,  $f_{0,4}$  and

$f_{0,5}$ , the theorem follows for  $p = 2, 3, 4, 5$ . The proof of Theorem 3.2 is very similar and hence omitted.

## CHAPTER FOUR A QUARTIC SPLINE

### Introduction and Statement of Theorems

Among the many beautiful properties of the complete cubic spline is the fact that for a given partition and function values, the cubic spline is obtained by solving a tridiagonally dominant system of equations. Unfortunately, when one uses higher order complete splines the bandwidth grows. In fact, for a  $2m$  times continuous spline of order  $2m+1$ , the bandwidth of the system of equations is  $2m+1$ . Furthermore the diagonal becomes less dominant as  $k$  increases.

It is natural then, to increase the order of the spline but preserve bandwidth. Ideally we would hope to increase the diagonal dominance and order of convergence. In this chapter we introduce a quartic  $C^{(2)}$  spline which gives  $O(h^5)$  rate approximation to a  $C^{(5)}$  function. The quartics are obtained by the solution of a tridiagonally dominant system. As desired, it is more diagonally dominant than the system associated with the complete cubic spline.

The main result of this chapter will be to give an exact error bound for the quartic spline discussed here. We first give the definition.

Let  $f$  be a real-valued function defined on  $[a, b]$ .  
Choose a partition  $\{x_i\}_{i=0}^k$  such that

$$a = x_0 < x_1 < \dots < x_k = b.$$

Let  $z_i = (x_{i-1} + x_i)/2$ , be the midpoint of  $[x_{i-1}, x_i]$  for  $i = 1, 2, \dots, k$  and for these  $i$  set  $h_{i-1} = x_i - x_{i-1}$ .

Definition 4.1 Given the function  $f$  and the partition  $\{x_i\}_{i=0}^k$ , we define a quartic spline  $s(x)$  such that

$$(4.1.1) \quad s(x) \in C^2[a, b] \cap P_4[x_{i-1}, x_i], \quad i = 1, 2, \dots, k;$$

(where  $P_4[x_{i-1}, x_i]$  denotes the functions which are quartics when restricted to  $[x_{i-1}, x_i]$ )

$$(4.1.2) \quad s(x_i) = f(x_i) \text{ for } i = 0, 1, \dots, k$$

$$s(z_i) = f(z_i) \text{ for } i = 1, 2, \dots, k;$$

and

$$(4.1.3) \quad s'(a) = f'(a) \text{ and } s'(b) = f'(b).$$

Lemma 4.1 Let  $f$  be a real-valued function defined on  $[a, b]$  and let  $\{x_i\}_{i=0}^k$  be a partition of  $[a, b]$ . A quartic spline  $s$  satisfies Equations (4.1.1) and (4.1.2) if and only if  $s$  satisfies the tridiagonal system of equations for  $i = 1, 2, \dots, k-1$

$$(4.1.4)$$

$$\begin{aligned} & -h_i s'(x_{i-1}) + 4(h_i + h_{i-1}) s'(x_i) - h_{i-1} s'(x_{i+1}) \\ & = -11 [(h_{i-1}/h_i) - (h_i/h_{i-1})] f(x_i) \\ & + 16 [(h_{i-1}/h_i) f(z_{i+1}) - (h_i/h_{i-1}) f(z_i)] \\ & - 5 [(h_{i-1}/h_i) f(x_{i+1}) - (h_i/h_{i-1}) f(x_{i-1})] \end{aligned}$$

where  $h_i = x_{i+1} - x_i$ .

We will give the proof later.

Assuming from (4.1.2) that  $f(x_i)$ ,  $i = 0, 1, \dots, k$  and  $f(z_i)$ ,  $i = 1, 2, \dots, k$  are known, then (4.1.4) is a system of  $k - 1$  equations in the unknown variables  $s'(x_i)$ ,  $i = 1, 2, \dots, k$ . If we impose the conditions of (4.1.3) that  $s'(x_0) = f'(a)$  and  $s'(x_k) = f'(b)$  are given, we have  $k-1$  unknowns and the  $k - 1$  diagonally dominant equations (4.1.4). Lemma 4.1 thus assures us that  $s'(x_i)$  can be uniquely determined for given conditions (4.1.1)-(4.1.3). As will be shown in the proof of the lemma, there is, on any given subinterval  $[x_i, x_{i+1}]$ , a unique quartic  $s_i(x)$  satisfying the five conditions

$$(4.1.5) \quad \begin{aligned} s_i(x_i) &= f(x_i) , & s_i(z_{i+1}) &= f(z_{i+1}) , \\ s_i(x_{i+1}) &= f(x_{i+1}) , & s'_i(x_i) &= s'(x_i) , \\ s'_i(x_{i+1}) &= s'(x_{i+1}) . \end{aligned}$$

Equations (4.1.4) are derived by imposing the conditions that

$$s_i''(x_i+) = s_{i-1}''(x_i-) \text{ for } i = 1, 2, \dots, k - 1 .$$

For  $i = 1, 2, \dots, k$ ,  $s_i(x)$  is thus the restriction of the spline  $s$  to  $[x_i, x_{i+1}]$ .

Summarizing, unique solution of (4.1.4) implies that  $s(x)$  is uniquely defined on each partition subinterval  $[x_i, x_{i+1}]$ ,  $i = 0, 1, \dots, k-1$ , which is to say, on all of  $[a, b]$ . We have shown

Corollary 4.1 The quartic spline of Definition 4.1 is, for a given partition  $\{x_i\}_{i=0}^k$  and function  $f$ , unique.

We now make the comparisons with the complete cubic spline more explicit. The system of equations

corresponding to (4.1.4) for the complete cubic spline has left-hand side

$$h_i s'(x_{i-1}) + 2(h_i + h_{i-1}) s'(x_i) + h_{i-1} s'(x_{i+1}) .$$

In comparison (4.1.4) is twice as diagonally dominant.

To interpolate the  $2k + 1$  function values  $f(x_i)$  and  $f(z_i)$  using our  $C^{(2)}$  quartic required solving the tridiagonal system of  $k - 1$  equations (4.1.4). As the cubic spline must match derivative and second derivative values at each interior function value, interpolation of the same  $2k + 1$  function values by the  $C^{(2)}$  cubic spline would entail solution of a system of  $2k - 1$  equations. In other words, the matrix equation to be solved for the quartic is only half as large as that required for the cubic.

We can now state the main theorem of this chapter. Given a partition  $\{x_i\}_{i=0}^k$  of  $[a, b]$ , denote

$$h = \max_{0 \leq i \leq k-1} h_i = \max_{0 \leq i \leq k-1} (x_{i+1} - x_i) .$$

For each  $x$  in  $[a, b]$ , there exists  $i$  such that

$$0 \leq i \leq k - 1 \text{ and } x_i \leq x \leq x_{i+1} . \text{ We set } t = (x - x_i)/h_i .$$

Theorem 4.1 Let  $f \in C^{(5)}[a, b]$  and let  $\{x_i\}_{i=0}^k$  be a partition of  $[a, b]$ . Let  $s(x)$  be the twice continuously differentiable spline corresponding to  $f$  and  $\{x_i\}_{i=0}^k$ , where  $s$  satisfies (4.1.1)-(4.1.3). Then

$$(4.1.6) \quad |f(x) - s(x)| \leq |c(t)| h^5 \max_{a \leq x \leq b} |f^{(5)}(x)| / 5!$$

where

$$c(t) = [3t^2(1-2t)(1-t)^2 + t(1-2t)(1-t)] / 6 .$$

Define

$$c_0 = \max_{0 \leq t \leq 1} |c(t)|$$

$$= \left( \sqrt{\frac{1}{4} - \frac{1}{\sqrt{30}}} \right) (1/5 + 2\sqrt{30}) / 6.$$

It follows that

$$(4.1.7) \quad |f(x) - s(x)| \leq c_0 h^5 \max_{a \leq x \leq b} |f^{(5)}(x)| / 5!.$$

Furthermore, neither  $|c(t)|$  nor  $c_0$  can be improved, as we can show by letting  $f = x^5/5!$  and letting  $k$  become arbitrarily large for an equally spaced partition. An approximate decimal expression for  $c_0$  is .0244482 and  $c_0/5!$  is approximately .000203818.

We will also show

$$(4.1.8) \quad |f'(x_i) - s'(x_i)|$$

$$\leq h^4 \max_{a \leq x \leq b} |f^{(5)}(x)| / 6!$$

and that this estimate is exact.

Related to Theorem 4.1 is the following conjecture.

Conjecture 4.1 Let  $f \in C^{(5)}[a,b]$  and let  $\{x_i\}_{i=0}^k$  be a partition of  $[a,b]$ . Let  $s(x)$  be the twice continuously differentiable spline corresponding to  $f$  and  $\{x_i\}_{i=0}^k$ , where  $s$  satisfies (4.1.1)-(4.1.3). Then

$$(4.1.9) \quad |f'(x) - s'(x)| \leq h^4 \max_{a \leq x \leq b} |f^{(5)}(x)| / 6!.$$

If Conjecture 4.1 holds, then the constant  $1/6!$  can not be improved. This conjecture has been verified numerically.

Remark 4.1 Given  $f \in C^5[a,b]$  and a partition  $\{x_i\}_{i=0}^k$  of  $[a,b]$ , let  $s$  be the quartic  $C^{(2)}$  spline satisfying (4.1.1)-(4.1.3). Then the supremum norm  $\|f^{(i)} - s^{(i)}\|$  is of order  $h^{5-i} \max_{a \leq x \leq b} |f^{(5)}(x)|$ ,  $i = 0, 1, 2$ .

Theorem 4.1 demonstrates that the quartic  $C^{(2)}$  spline gives the best possible order of approximation to functions from the smooth class  $C^{(5)}$ . We next discuss interpolation to the much less smooth class of functions which are merely continuous on  $[a,b]$ . As  $f'(a)$  and  $f'(b)$  are not necessarily defined, we consider the quartic  $C^{(2)}$  spline satisfying (4.1.1) and (4.1.2) with boundary conditions

$$(4.1.10) \quad s'(a) = s'(b) = 0.$$

Denote  $\omega(f,h) =: \sup_{|x-y| \leq h} |f(x) - f(y)|$ .

Theorem 4.2 Let  $f \in C[a,b]$ . If  $\{x_i\}_{i=0}^k$  is the partition of equally spaced knots, then for  $x_i \leq x \leq z_{i+1} = (x_i + x_{i+1})/2$  and  $t = (x - x_i)/h_i$ ,  $i = 0, 1, \dots, k-1$ , we have

$$(4.1.11) \quad |f(x) - s(x)| \leq c(t) \omega(f,h); \quad 0 \leq t \leq 1/2$$

and for  $z_{i+1} \leq x \leq x_{i+1}$ , or  $1/2 \leq t \leq 1$

$$(4.1.12) \quad |f(x) - s(x)| \leq c(1-t) \omega(f,h)$$

where  $c(t) = \{1 + (13/3)t - 3t^2 - (58/3)t^3 + 16t^4\}$ .

Note that  $\max_{0 \leq t \leq 1/2} c(t)$  is approximately equal to 1.6572.

The bound of the preceding theorem is only valid for equally spaced knots. For arbitrary partitions we can not give a bound of this same form. However, if

$$m =: \max_{0 \leq i \leq k-1} h_i / \min_{0 \leq i \leq k-1} h_i,$$

we have the following theorem.

Theorem 4.1.3 Let  $f(x) \in C[a,b]$ , and let  $s$  be the  $C^{(2)}$  quartic spline satisfying (4.1.1), (4.1.2), and (4.1.9). Then for  $x_i \leq x \leq x_{i+1}$ , and  $i = 0, 1, 2, \dots, k-1$ , (i.e., for  $0 \leq t \leq 1/2$  with  $t = (x - x_i)/h_i$ )

$$(4.1.13) \quad |f(x) - s(x)| \leq c_1(t) \omega(f, h) ,$$

and for  $x_{i+1} \leq x \leq x_{i+2}$ , i.e.  $1/2 \leq t \leq 1$ ,

$$(4.1.14) \quad |f(x) - s(x)| \leq c_1(1-t) \omega(f, h)$$

where

$$c_1(t) = [1 + 10t^2 - 28t^3 + 16t^4] \\ + (8/3)[m^2 + m] [t(1-2t)(1-t)] .$$

Theorems 4.2 and 4.3 indicate that for suitable partitions the quartic  $C^2$  spline can provide acceptable approximations to functions which are merely continuous on  $[a,b]$ .

#### Proof of Lemma 4.1

We first give an expression for the unique quartic matching function and derivative values at endpoints and function values at the midpoint. Specifically, let  $f$  be a real-valued function defined on  $[0,1]$ , and differentiable at 0 and 1. Let

(4.2.1)

$$P_1(x) = 1 - 11x^2 + 18x^3 - 8x^4 = (1-2x)(1-x)^2(1+4x) ,$$

$$P_2(x) = 16x^2 - 32x^3 + 16x^4 = 16x^2(1-x)^2 ,$$

$$P_3(x) = -5x^2 + 14x^3 - 8x^4 = -(1-2x)x^2[1+4(1-x)] ,$$

$$P_4(x) = x - 4x^2 + 5x^3 - 2x^4 = x(1-2x)(1-x)^2 ,$$

$$P_5(x) = x^2 - 3x^3 + 2x^4 = x^2(1-2x)(1-x) .$$

Then

$$(4.2.2) \quad L[f, x] = P_1(x)f(0) + P_2(x)f(1/2) + P_3(x)f(1) \\ + P_4(x)f'(0) + P_5(x)f'(1)$$

is the unique quartic satisfying

$$(4.2.3) \quad L[f, 0] = f(0) , \quad L[f, 1/2] = f(1/2) ,$$

$$L[f, 1] = f(1) , \quad L'[f, 0] = f'(0) , \quad L'[f, 1] = f'(1) .$$

$L$  is a linear functional and a projection. If  $f$  is a polynomial of degree four or less, then  $L[f, x] = f(x)$ . In the future calculations we will need the following facts about the quartics  $P_i$ ,

$$(4.2.4) \quad \begin{array}{ll} P_1''(0) = -22 & P_1''(1) = -10 \\ P_2''(0) = 32 & P_2''(1) = 32 \\ P_3''(0) = -10 & P_3''(1) = -22 \\ P_4''(0) = -8 & P_4''(1) = -2 \\ P_5''(0) = 2 & P_5''(1) = 8 . \end{array}$$

Let  $z_{i+1} = (x_i + x_{i+1})/2$ . On the interval  $[x_i, x_{i+1}]$ , the unique quartic  $L_i[f, x]$  interpolating  $f(x_i)$ ,  $f'(x_i)$ ,  $f(z_{i+1})$ ,  $f(x_{i+1})$ , and  $f'(x_{i+1})$  can be expressed in terms of  $P_i$ . In fact, let  $t = (x - x_i)/h_i$  where  $h_i = x_{i+1} - x_i$ . Then

$$(4.2.5) \quad L_i[f, x] = f(x_i) P_1(t) + f(z_{i+1}) P_2(t) \\ + f(x_{i+1}) P_3(t) + h_i f'(x_i) P_4(t) \\ + h_i f'(x_{i+1}) P_5(t) .$$

Let  $s$  be the quartic spline of Definition 4.1 corresponding to  $f$  and the given partition. Then the restriction  $s_i(x)$  of  $s$  to  $[x_i, x_{i+1}]$  is a quartic. Hence

$L_i[s, x] = s_i(x)$ . Using the facts that  $s(x_i) = f(x_i)$ ,  $s(z_{i+1}) = f(z_{i+1})$ , and  $f(x_{i+1}) = s(x_{i+1})$  in (4.2.5) we have

$$(4.2.6) \quad s_i(x) = f(x_i) P_1(t) + f(z_{i+1}) P_2(t) \\ + f(x_{i+1}) P_3(t) + h_i s'(x_i) P_4(t) \\ + h_i s'(x_{i+1}) P_5(t), \quad t = (x - x_i)/h_i.$$

In order that  $s$  be twice continuously differentiable, we must satisfy

$$(4.2.7) \quad s_i''(x_i+) = s_{i-1}''(x_i-)$$

where  $s_i$  is the restriction of  $s$  to  $[x_i, x_{i+1}]$  and  $s_{i-1}$  is the restriction of  $s$  to  $[x_{i-1}, x_i]$ . Differentiating

(4.2.6) twice we have

$$(4.2.8) \quad s''(x_i+) = \frac{1}{h_i^2} \{ f(x_i) P_1''(0) + f(z_{i+1}) P_2''(0) \\ + f(x_{i+1}) P_3''(0) + h_i s'(x_i) P_4''(0) \\ + h_i s'(x_{i+1}) P_5''(0) \}.$$

Similarly, from rewriting (4.2.6) for the interval

$[x_{i-1}, x_i]$ , we have

$$(4.2.9) \quad s''(x_i-) = \frac{1}{h_{i-1}^2} \{ f(x_{i-1}) P_1''(1) + f(z_i) P_2''(1) \\ + f(x_i) P_3''(1) \\ + h_{i-1} s'(x_{i-1}) P_4''(1) \\ + h_{i-1} s'(x_i) P_5''(1) \}.$$

Setting  $s''(x_i+) = s''(x_i-)$  by equating (4.2.8) and (4.2.9) and using  $P_1''(0)$  and  $P_1''(1)$  from (4.2.4), we have

$$\begin{aligned}
(4.2.10) \quad & \{-22 f(x_i) + 32 f(z_{i+1}) - 10 f(x_{i+1}) \\
& \quad - 8 h_i s'(x_i) + 2 h_i s'(x_{i+1})\} / h_i^2 \\
& = \{-10 f(x_{i-1}) + 32 f(z_i) - 22 f(x_i) \\
& \quad - 2 h_{i-1} s'(x_{i-1}) + 8 h_{i-1} s'(x_i)\} / h_{i-1}^2 .
\end{aligned}$$

Factoring two, multiplying by  $h_i h_{i-1}$ , and putting the known function values on the right hand side, we have

$$\begin{aligned}
(4.2.11) \quad & -h_i s'(x_{i-1}) + 4(h_i + h_{i-1}) s'(x_i) - h_{i-1} s'(x_{i+1}) \\
& = -11 [(h_{i-1}/h_i) - (h_i/h_{i-1})] f(x_i) \\
& \quad + 16 [(h_{i-1}/h_i) f(z_{i+1}) - (h_i/h_{i-1}) f(z_i)] \\
& \quad - 5 [(h_{i-1}/h_i) f(x_{i+1}) - (h_i/h_{i-1}) f(x_{i-1})] ,
\end{aligned}$$

which is the desired system of equations (4.1.4). Having established Lemma 4.1, we next turn to a proof of Theorem 4.1.

#### Proof of Theorem 4.1

Our method of proof is to establish a pointwise bound. As in the proof of Lemma 4.1, let  $L_i[f, x]$  be the unique quartic agreeing with  $f(x_i)$ ,  $f(x_{i+1})$ ,  $f(z_{i+1})$ ,  $f'(x_i)$ , and  $f'(x_{i+1})$ , and let  $s$  be the twice continuous quartic spline corresponding to  $f$  and Equations (4.1.1) to (4.1.3) on the partition  $\{x_i\}_{i=0}^k$ . Then for  $x_i \leq x \leq x_{i+1}$ , we have

$$\begin{aligned}
(4.3.1) \quad |f(x) - s(x)| & \leq |f(x) - L_i[f, x]| \\
& \quad + |L_i[f, x] - s(x)| .
\end{aligned}$$

Assume that  $f \in C^{(5)}[a, b]$ . By a proof attributed to Cauchy, we know that

(4.3.2)  $|f(x) - L_i[f, x]| \leq (h_i^5/5!) |t^2(1/2-t)(1-t)^2| U$   
 where  $t = (x-x_i)/h_i$  and  $U$  is the maximum of  $|f^{(5)}(x)|$  on  $[x_i, x_{i+1}]$ . Equation (4.1.9) gives a pointwise bound for  $|f(x) - L_i[f, x]|$ .

Let  $i$  be arbitrary and  $x_i \leq x \leq x_{i+1}$ . We next turn our attention to deriving a similar bound for

$|L_i[f, x] - s(x)| = |L_i[f, x] - s_i(x)|$ . Subtracting (4.2.6) from (4.2.5) gives

$$(4.3.3) \quad L_i[f, x] - s_i(x) = h_i [f'(x_i) - s'(x_i)] P_4(t) + h_i [f'(x_{i+1}) - s'(x_{i+1})] P_5(t).$$

Denoting

$$(4.3.4) \quad e'(x_i) = f'(x_i) - s'(x_i),$$

then we have from (4.2.5),

$$(4.3.5) \quad |L_i[f, x] - s_i(x)| \leq h_i \max\{|e'(x_i)|, |e'(x_{i+1})|\} \{|P_4(t)| + |P_5(t)|\}.$$

As  $P_4(t) = t(1-2t)(1-t)^2$  and  $P_5(t) = t^2(1-2t)(1-t)$  are both positive for  $0 \leq t \leq 1/2$  and both negative for  $1/2 \leq t \leq 1$ ,  $|P_4(t)| + |P_5(t)| = |P_4(t) + P_5(t)|$  for  $0 \leq t \leq 1$ . Then for  $x_i \leq x \leq x_{i+1}$ , we have

$$(4.3.6) \quad |L_i[f, x] - s(x)| \leq h_i \max\{|e'(x_i)|, |e'(x_{i+1})|\} |t(1-2t)(1-t)|.$$

Redefine  $L$  so that its restriction to  $[x_i, x_{i+1}]$  is  $L_i$  for each  $i$ ,  $i = 0, 1, \dots, k-1$ . Choose  $i$  so that  $|e'(x_i)|$  is maximal. We then have for all  $a \leq x \leq b$ ,

$$(4.3.7) \quad |L[f, x] - s(x)| \leq h |e'(x_i)| |t(1/2-t)(1-t)|$$

where  $h = \max_{0 \leq j \leq k-1} h_j$  is the maximal subinterval length and where on each subinterval  $[x_j, x_{j+1}]$ ,  $0 \leq j \leq k-1$ , we define  $t = (x - x_j)/h_j$ .

The next task is to bound  $|e'(x_i)|$ . From both sides of (4.1.4) we subtract

$$-h_i f'(x_{i-1}) + 4(h_i + h_{i-1})f'(x_i) - h_{i-1}f'(x_{i+1}) ,$$

thereby defining a functional  $B_0(f)$

$$\begin{aligned} (4.3.8) \quad & h_i e'(x_{i-1}) - 4(h_i + h_{i-1})e'(x_i) + h_{i-1}e'(x_{i+1}) \\ & = h_i f'(x_{i-1}) - 4(h_i + h_{i-1})f'(x_i) + h_{i-1}f'(x_{i+1}) \\ & \quad - 11 [(h_{i-1}/h_i) - (h_i/h_{i-1})] f(x_i) \\ & \quad + 16 [(h_{i-1}/h_i) f(x_{i+1}) - (h_i/h_{i-1}) f(x_{i-1})] \\ & \quad - 5 [(h_{i-1}/h_i) f(x_{i+1}) - (h_i/h_{i-1}) f(x_{i-1})] \\ & =: B_0(f) . \end{aligned}$$

The linear functional  $B_0(f)$  is identically equal to zero when  $f$  is a polynomial of degree four or less, as can be directly verified. (The arithmetic of verification is simplest if one takes  $x_{i-1} = -h_{i-1}$ ,  $x_i = 0$ ,  $x_{i+1} = h_i$  and checks the monomials  $1, x, x^2, x^3$ , and  $x^4$ ).

We have chosen  $i$  so that  $|e'(x_i)|$  attains its maximum value. As

$$\begin{aligned} & 4(h_i + h_{i-1})e'(x_i) \\ & = -B_0(f) + h_i e'(x_{i-1}) + h_{i-1}e'(x_{i+1}), \end{aligned}$$

it follows that

$$\begin{aligned} |4(h_i + h_{i-1})| & \leq |B_0(f)| + |h_i e'(x_{i-1})| \\ & \quad + |h_{i-1}e'(x_{i+1})| \\ & \leq |B_0(f)| + |(h_i + h_{i-1})e'(x_i)| . \end{aligned}$$

Hence

$$|3(h_i + h_{i-1})e'(x_i)| \leq |B_0(f)|$$

and

$$(4.3.9) \quad |e'(x_i)| \leq |B_0(f)| / [3(h_i + h_{i-1})] .$$

As  $B_0(f)$  is a linear functional which is zero for polynomials of degree four or less, we can apply the Peano theorem to get

$$(4.3.10) \quad B_0(f) = \int_{x_{i-1}}^{x_{i+1}} B_0[(x-y)_+^4] f^{(5)}(y) dy / 4! .$$

From (4.3.10) follows

$$(4.3.11) \quad |B_0(f)| \leq \int_{x_{i-1}}^{x_{i+1}} |B_0[(x-y)_+^4]| dy U_i / 4! .$$

where  $U_i$  is the maximum of  $|f^{(5)}|$  on  $[x_{i-1}, x_{i+1}]$ .

For  $x_{i-1} \leq y \leq x_{i+1}$ ,  $B_0[(x-y)_+^4]$  takes the form

$$(4.3.12) \quad B_0[(x-y)_+^4] = \\ - 16(h_i + h_{i-1}) (x_i - y)_+^3 + 4h_{i-1} (x_{i+1} - y)^3 \\ - 11 [(h_{i-1}/h_i) - (h_i/h_{i-1})] (x_i - y)_+^4 \\ + 16 [(h_{i-1}/h_i) (z_{i+1} - y)_+^4 - (h_i/h_{i-1}) (z_i - y)_+^4 \\ - 5 (h_{i-1}/h_i) (x_{i+1} - y)^4 .$$

In order to evaluate the integral of (4.3.11) we need to know the sign behavior of  $B_0[(x-y)_+^4]$ . We rewrite (4.3.12) in a form which shows its symmetry about  $x_i$ .

$$(4.3.13) \quad B_0[(x-y)_+^4] = \\ (h_i/h_{i-1}) [-5(x_i - y) + h_{i-1}] [(x_i - y) - h_{i-1}]^3 , \\ \text{for } x_{i-1} \leq y \leq z_i$$

$$(h_i/h_{i-1}) (x_i - y)^2 [11(x_i - y)^2 - 16h_{i-1}(x_i - y) + 6h_{i-1}^2] ,$$

$$\text{for } z_i \leq y \leq x_i$$

$$(h_{i-1}/h_i) (x_i - y)^2 [11(x_i - y)^2 + 16h_i(x_i - y) + 6h_i^2] ,$$

$$\text{for } x_i \leq y \leq z_{i+1}$$

$$(h_{i-1}/h_i) [-5(x_i - y) - h_i] [(x_i - y) + h_i]^3 ,$$

$$\text{for } z_{i+1} \leq y \leq x_{i+1} .$$

As the expression (4.3.13) has factors which are at most quadratic it is fairly easy to determine to determine the sign of  $B_0[(x-z)_+^4]$ . In fact,  $B_0[(x-z)_+^4]$  is nonnegative for  $z_{i-1} \leq y \leq x_{i+1}$ . Evaluation of (4.3.11) is then straightforward. The term by term integration of (4.3.13) gives

$$(4.3.14) \quad \int_{x_{i-1}}^{x_{i+1}} |B_0[(x-y)_+^4]| dy = h_i h_{i-1} [h_{i-1}^3 + h_i^3] / 10 .$$

From Equation (4.3.11) we conclude that

$$(4.3.15) \quad |B_0(f)| \leq U_i h_i h_{i-1} [h_{i-1}^3 + h_i^3] / [2(5!)] .$$

From (4.3.9) it is then evident that

$$(4.3.16) \quad |e'(x_j)| \leq U_i h_i h_{i-1} [h_{i-1}^3 + h_i^3] / [(6!)(h_i + h_{i-1})]$$

for  $j = 1, 2, \dots, k-1$ . As

$$[h_{i-1}^3 + h_i^3] / [h_i + h_{i-1}] \leq \max \{h_i^2, h_{i-1}^2\} ,$$

and as

$$U_i \leq U ,$$

it follows that

$$(4.3.17) \quad \max |e'(x_j)| \leq \max\{h_i^4, h_{i-1}^4\} U / (6!) .$$

This is the desired bound on  $|e'(x_i)|$ .

Applying it in (4.3.7) we have

$$(4.3.18) \quad |L[f, x] - s(x)| \leq h^5 |t(1-2t)(1-t)| U / (6!) .$$

From (4.3.2) follows

$$(4.3.19) \quad |f(x) - L[f, x]| \leq h^5 |t^2(t-1/2)(1-t)^2| U / 5!$$

where  $L$  restricted to  $[x_i, x_{i+1}]$  is defined as  $L_i[f, x]$  and where  $h$  is the maximum of  $h_i$ .

We can now combine the bounds on  $|f(x) - L[f, x]|$  and  $|L[f, x] - s(x)|$ . From (4.3.19) and (4.3.18), we have

$$(4.3.20) \quad |f(x) - s(x)| \leq h^5 |c(t)| U / 5!$$

where

$$\begin{aligned} |c(t)| &= |3t^2(1-2t)((1-t)^2| + |t(1-2t)(1-t)| / 6 \\ &= |3t^2(1-2t)(1-t)^2 + t(1-2t)(1-t)| / 6 \end{aligned}$$

and

$$c(t) = [3t(1-t) + 1] [t(1-2t)(1-t)] / 6 .$$

Then

$$(4.3.21) \quad c_0 = \max_{0 \leq t \leq 1} |c(t)| .$$

To verify (4.3.21), note that

$$\begin{aligned} (4.3.22) \quad 6c'(t) &= -30t^2(t-1)^2 + 1 \\ &= -30 [(t - 1/2) + 1/2]^2 [(t - 1/2) - 1/2]^2 + 1 \\ &= -30 [(t - 1/2)^2 - 1/4]^2 + 1 . \end{aligned}$$

For  $0 \leq t \leq 1$ , the roots of  $c'(t)$  are

$$(4.3.23) \quad t = 1/2 \pm \sqrt{1/4 - 1/\sqrt{30}} .$$

Evaluating  $c(t)$  at the roots of  $c'(t)$ , we get

$$(4.3.24) \quad c_0 = (\sqrt{1/4 - 1/\sqrt{30}}) (1/5 + 2/\sqrt{30}) .$$

We have shown the so-called "direct" part of the proof, that Equation (4.1.7) holds for  $c_0$ . It remains to be shown that the theorem holds for no smaller  $c_0$ .

In fact, given  $c < c_0$ , we can produce a function  $f$  and a partition  $\{x_i\}_{i=0}^k$  of  $[-1, 1]$  such that

$$(4.3.25) \quad \max_{-1 \leq x \leq 1} |f(x) - s(x)| > c h^5 \max_{-1 \leq x \leq 1} |f^{(5)}(x)|/5!.$$

Often, when polynomial interpolation of degree  $n$  is considered, the worst error is attained by a polynomial of degree  $n + 1$ . As  $s$  is a quartic spline, it is natural to try  $f(x) = x^5/5!$  as a possible worst function. A particularly pleasant feature of the trial worst function  $f$  is that it has fifth derivative identically equal to one.

For  $x_i \leq x \leq x_{i+1}$ , we have by the Cauchy formula

$$(4.3.26) \quad x^5/5! - L_i[x^5/5!, x] \\ = h_i^5 [t^2(t-1/2)(t-1)^2] / 5!.$$

Furthermore, for equally spaced knots  $x_{i-1}$ ,  $x_i$ ,  $x_{i+1}$ , we can calculate

$$(4.3.27) \quad B_0(x^5/5!) = h_i^5 / 5!.$$

If  $e'(x_{i-1}) = e'(x_i) = e'(x_{i+1})$ , we have from (4.3.8)

$$(4.3.28) \quad e'(x_i) = -B_0(x^5) / 6 = -h_i^4 / 6!.$$

Equation (4.3.3) then becomes for  $f(x) = x^5/5!$

$$(4.3.29) \quad L_i[f, x] - s(x) = -h_i h_i^4 \{P_4(t) + P_5(t)\}/6! \\ = h_i^5 [t(2t-1)(1-t)]/6!.$$

Combining Equations (4.3.26) and (4.3.29), we have, for  $x_i \leq x \leq x_{i+1}$ ,

$$(4.3.30) \quad f(x) - s(x) = h_1^5 \{t(t-1/2)(1-t)/3 + t^2(t-1/2)((1-t)^2) / 5!\}$$

As (4.3.30) gives, after taking its absolute value, precisely our pointwise bound  $|c(t)|$  of (4.3.20), we will have attained  $c_0$ , provided only that  $h_1 = h$ , and as mentioned above,

$$(4.3.31) \quad e'(x_i) = e'(x_{i+1}) = e'(x_{i-1}) = -h^4/6!$$

In order that  $h_1 = h$ , we take the knots to be equally spaced. Attaining (4.3.31) is not so easy. In fact it is attained only in the limit. The difficulty is the boundary conditions  $e'(x_0) = e'(x_k) = 0$ . We can show, however, that as one moves many subintervals away from the boundaries,  $e'(x_i)$  goes to  $-h^4/6!$ .

Explicitly, let  $\{x_i\}_{i=0}^k$  be the partition dividing  $[-1,1]$  into  $k$  equal subintervals; in this case,  $h = h_1 = 2/k$ . For  $i = 1, 2, \dots, k-1$ , and  $f = x^5/5!$ , we have  $B_0(f)$  defined on  $[x_{i-1}, x_{i+1}]$  and

$$(4.3.32) \quad B_0(f)/h = h^4/5! = e'(x_{i-1}) - 8e'(x_i) + e'(x_{i+1})$$

We wish to apply (4.3.32) inductively to move away from the end conditions  $e'(-1) = e'(1) = 0$ . In order to do so we must establish that  $e'(x_i) \leq 0$  for  $0 \leq i \leq k$ . We reason by contradiction.

Let  $1 \leq i \leq k-1$ . Suppose  $e'(x_i) > 0$ . Then

$$\begin{aligned} e'(x_{i-1}) + e'(x_{i+1}) &\geq \\ &= e'(x_{i-1}) - 8e'(x_i) + e'(x_{i+1}) \\ &\geq h^4/5! \end{aligned}$$

Hence

$$\max\{ |e'(x_{i-1})|, |e'(x_{i+1})| \} \geq h^4/[2(5!)] ,$$

contradicting the fact (4.3.17) that

$$h^4/6! \geq \max\{ |e'(x_{i-1})|, |e'(x_{i+1})| \} .$$

We have shown by assuming the contrary that

$$e'(x_i) \leq 0 \text{ for } i = 1, 2, \dots, k-1 .$$

Condition (4.1.3) is that  $e'(x_0)$  and  $e'(x_k)$  are zero. Thus

$$(4.3.33) \quad e'(x_i) \leq 0 \text{ for } i = 0, 1, \dots, k .$$

Applying (4.3.32) again we have for  $i = 1, 2, \dots, k-1$

$$8e'(x_i) = -h^4/5! + e'(x_{i-1}) + e'(x_{i+1}) .$$

As  $e'(x_{i-1}), e'(x_{i+1}) \leq 0$ , this implies that

$$8e'(x_i) \leq -h^4/5!$$

and

$$(4.3.34) \quad e'(x_i) \leq -h^4/[8(5!)] .$$

Similarly, for  $i = 2, 3, \dots, k-2$ , we have

$$8e'(x_i) = -h^4/5! + e'(x_{i-1}) + e'(x_{i+1}) ,$$

and hence by (4.3.34),

$$\begin{aligned} e'(x_i) &\leq -h^4/5! - h^4/[8(5!)] - h^4/[8(5!)] \\ &= -(1 + 1/4) h^4/[8(5!)] . \end{aligned}$$

Inductively, for  $i = j$  to  $i = k-j$ , we will have

$$e'(x_i) \leq -\{1 + 1/4 + 1/4^2 + \dots + 1/4^{j-1}\}h^4/[8(5!)] .$$

The harmonic series  $(1 + 1/4 + 1/4^2 + \dots)$  is equal to  $1/(1 - 1/4)$  or  $4/3$ . Thus, in the limit as  $i, k$ , and  $j$  go to infinity, we have

$$(4.3.35) \quad e'(x_i) \leq -(1/8)(4/3)h^4/5! = -h^4/6! .$$

We already know from (4.3.17) that  $|e'(x_i)| \leq h^4/6!$ . Thus for  $k > 2j + 1$ , and  $k - j > i > j$  as  $j$  goes to infinity,

we have

$$(4.3.36) \quad e'(x_i) \text{ goes to } -h^4/6! .$$

In the sense of (4.3.36), (4.3.31) is satisfied. Then, as  $e'(x_i)$  goes to  $-h^4/6!$ ,  $x^5/5! - s(x)$  goes uniformly to the expression of (4.3.30) and (4.3.20) with  $h = h_i$ . It follows that the expression of (4.3.20) cannot be improved further. In fact we have shown that  $|c(t)|$  offers a pointwise exact bound, and its maximum  $c_0$  is the exact norm bound.

#### Proof of Theorem 4.2

We know from (4.3.1) that for  $x_i \leq x \leq x_{i+1}$ ,

$$(4.4.1) \quad \begin{aligned} s(x) - f(x) = & P_1(t) f(x_i) + P_2(t) f(z_{i+1}) \\ & + P_3(t) f(x_{i+1}) + h_i P_4(t) s'(x_i) \\ & + h_i P_5(t) s'(x_{i+1}) - f(x) . \end{aligned}$$

It is easily verified that  $P_1(t) + P_2(t) + P_3(t) = 1$ .

Thus

$$(4.4.2) \quad \begin{aligned} s(x) - f(x) = & P_1(t) [f(x_i) - f(x)] \\ & + P_2(t) [f(z_{i+1}) - f(x)] \\ & + P_3(t) [f(x_{i+1}) - f(x)] \\ & + h_i s'(x_i) P_4(t) \\ & + h_i s'(x_{i+1}) P_5(t) . \end{aligned}$$

Each of the first three terms on the right hand side can be bounded in absolute value by  $(f, h)$ . We also must bound the last two terms. For equally spaced knots  $h = h_i = h_{i-1}$ ; equation (4.1.4) reduces to

$$\begin{aligned}
 (4.4.3) \quad & -h s'(x_{i-1}) + 8h s'(x_i) - h s'(x_{i+1}) \\
 & = 16 [f(z_{i+1}) - f(z_i)] \\
 & \quad - 5 [f(x_{i+1}) - f(x_{i-1})] .
 \end{aligned}$$

Assume  $s'(x_i)$  is maximal in absolute value. Then

$$\begin{aligned}
 |6h s'(x_i)| & \leq 16 |f(z_{i+1}) - f(z_i)| \\
 & \quad + 5 |f(x_{i+1}) - f(x_i)| \\
 & \quad + 5 |f(x_i) - f(x_{i-1})| \\
 & \leq 26 \omega(f, h) ,
 \end{aligned}$$

and hence

$$(4.4.4) \quad |h s'(x_i)| \leq (13/3) \omega(f, h) .$$

Combining (4.4.2) and (4.4.4), we have

$$\begin{aligned}
 (4.4.5) \quad |s(x) - f(x)| & \leq \{ |P_1(t)| + |P_2(t)| + |P_3(t)| \\
 & \quad + (13/3) |P_4(t)| + (13/3) |P_4(t)| \} \omega(f, h) .
 \end{aligned}$$

For  $0 \leq t \leq 1/2$ ,

$$\begin{aligned}
 (4.4.6) \quad & |P_1(t)| + |P_2(t)| + |P_3(t)| \\
 & \quad + (13/3) |P_4(t)| + (13/3) |P_4(t)| \\
 & = P_1(t) + P_2(t) - P_3(t) + \\
 & \quad (13/3)P_4(t) + (13/3)P_5(t) \\
 & = 1 + (13/3)t - 3t^2 - (58/3)t^3 + 16t^4 .
 \end{aligned}$$

We have shown the theorem for  $0 \leq t \leq 1/2$ . The argument for  $1/2 \leq t \leq 1$  is symmetric.

Proof of Theorem 4.3 We are considering now the case in which knots are no longer assumed to be equal. We assume that the ratio of the longest subinterval to the shortest is less than  $m$ . Equation (4.4.2) still applies. Again we choose  $i$  so that  $|s'(x_i)|$  is maximal. From (4.1.4) we now have

$$\begin{aligned}
(4.4.7) \quad & -h_i s'(x_{i-1}) + 4(h_i + h_{i-1}) s'(x_i) - h_{i-1} s'(x_{i+1}) \\
& = -11 [(h_{i-1}/h_i) - (h_i/h_{i-1})] f(x_i) \\
& \quad + 16 [(h_{i-1}/h_i) f(z_{i+1}) - (h_i/h_{i-1}) f(z_i)] \\
& \quad - 5 [(h_{i-1}/h_i) f(x_{i+1}) - (h_i/h_{i-1}) f(x_{i-1})] \\
& = 11[h_i/h_{i-1}] [f(x_i) - f(z_i)] \\
& \quad + 5[h_i/h_{i-1}] [-f(z_i) + f(x_{i-1})] \\
& \quad + 11[h_{i-1}/h_i] [f(z_{i+1}) - f(x_i)] \\
& \quad + 5[h_{i-1}/h_i] [f(z_{i+1}) - f(x_{i+1})] .
\end{aligned}$$

Then

$$\begin{aligned}
3(h_i + h_{i-1}) |s'(x_i)| & \leq 16 \frac{h_i}{h_{i-1}} \omega(f, h_{i-1}/2) \\
& \quad + 16 \frac{h_{i-1}}{h_i} \omega(f, h_i/2)
\end{aligned}$$

and

$$|s'(x_i)| \leq (16/3) \frac{[h_i/h_{i-1} + h_{i-1}/h_i] \omega(f, h)}{h_i + h_{i-1}}$$

where  $h = \max_{0 \leq i \leq k-1} h_i$ .

Then for any given  $j$ ,  $0 \leq j \leq k-1$ , and

$m = \max\{h_i\}/\{\min_i\}$ ,  $i = 0, 1, \dots, k-1$ , we have

$$\begin{aligned}
(4.4.8) \quad & \max\{|h_j s'(x_j)|, |h_j s'(x_{j+1})|\} \\
& \leq (16/3) \frac{h}{2\min\{h_i\}} (m + 1/m) \omega(f, h) \\
& \leq (8/3) (m^2 + m) \omega(f, h) .
\end{aligned}$$

Substituting (4.4.8) into (4.4.2) yields the result of the theorem.

## CHAPTER FIVE IMPROVED ERROR BOUNDS FOR THE PARABOLIC SPLINE

### Introduction and Statement of Theorems

The quartic splines of Chapter Four share and improve many of the properties of the complete cubic spline. To insure a good approximation to a given continuous function, we must make the largest subinterval of a partition small. Unfortunately, we must also pose some additional restrictions on the partition. For instance, in Theorem 4.3, the norm of the error depends not only on the length  $h$  of the largest subinterval but also on the ratio  $m$  of the largest to smallest length subinterval. Similar additional restrictions must be made for the cubic spline.

In this chapter, we will discuss a spline operator for which the norm of the approximation error goes to zero with the length of the largest subinterval, for any partition and any continuous periodic function. This spline is the piecewise parabolic spline introduced by Marsden and discussed in Chapter One. Its properties are summarized in Equations (1.6.1) to (1.6.7).

As Marsden points out, many of the bounds he gives can be sharpened. The main result of this chapter will be to accomplish this sharpening. While many of the bounds

given here may still not be exact, at least one of them is, and in fact is even pointwise exact. In other cases we can reduce the known bounds by a factor of more than two.

The results given here thus enable one to compare the error of the Marsden spline to the error of other spline interpolation processes. Specifically, future work on the cubic spline interpolant should shed light on the validity of Marsden's conjecture that the parabolic spline offers better approximation than the cubic spline when functions of the classes  $C^{(1)}$ ,  $C^{(2)}$ , and  $C^{(3)}$  are considered.

We first recapitulate the properties of the parabolic spline. Let

$$f \in C[a,b] , \quad f(a) = f(b) ,$$

$$||f|| = \sup \{ |f(x)| : a \leq x \leq b \}$$

such that  $f$  is extended periodically with period  $b - a$ .

A function  $s(x)$  is defined to be a periodic quadratic spline interpolant associated with  $f$  and a partition  $\{x_i\}_{i=0}^k$  if

(5.1.1)

a)  $s(x)$  is a quadratic expression on each  $(x_{i-1}, x_i)$  ;

b)  $s(x) \in C'[a,b]$  ;

c)  $s(a) = s(b)$  ,  $s'(a) = s'(b)$  ;

d)  $s(z_i) = f(z_i)$   $i = 1, 2, \dots, k$

where  $z_{i+1} = (x_{i+1} + x_i)/2$ .

The following theorem is due to Marsden [1974] and was given in Chapter One as Theorem 1.13.

Theorem Let  $\{x_i\}_{i=0}^k$  be a partition of  $[a, b]$ ,  $f(x)$  be a continuous function of period  $b - a$ , and  $s(x)$  be the periodic quadratic spline interpolant associated with  $f$  and  $\{x_i\}_{i=0}^k$ . Then

$$(5.1.2) \quad \begin{aligned} ||s_i|| &\leq 2 ||f||, & ||s|| &\leq 2 ||f||; \\ ||e_i|| &\leq 2 \omega(f, h/2); \\ ||e|| &\leq 3 \omega(f, h/2). \end{aligned}$$

(where  $s_i = s(x_i)$  and  $e_i = y(x_i) - s(x_i)$ ).

The constant 2 which appears in the first of the above equations can not, in general, be decreased.

For continuous functions to be "well-approximated" by the spline  $s$ , Equations (5.1.2) show that the only requirement for the partition is that the length  $h$  of the largest subinterval be small enough that the modulus of continuity of  $f$  be small.

Concerning  $s$ , we can prove the following results. These are analogous to the results of Marsden given above as Theorems 1.14 to 1.16 and improve upon the bounds he derived.

Theorem 5.1 Let  $f$  and  $f'$  be continuous functions of period  $b - a$ . Then

$$(5.1.3) \quad ||e(x)|| \leq c_{0,1} h ||f'||,$$

where  $a_0 = 2/3 - \sqrt{13}/6$  and  $c_{0,1} = 1 + a_0 - 8a_0^2 + 4a_0^3$  or  $c_{0,1}$  is approximately 1.0323. The analogous constant from Marsden was  $5/4$ .

Theorem 5.2 Let  $f$ ,  $f'$ , and  $f''$  be continuous functions of period  $b - a$ . Then

$$(5.1.4) \quad ||e|| \leq (1/6) h^2 ||f''|| ,$$

$$(5.1.5) \quad ||e_i'|| \leq (9/16) h ||f''|| ,$$

$$(5.1.6) \quad ||e'|| \leq (17/16) h ||f''|| .$$

(Marsden's constant for (5.1.4) was  $5/8$ , while in (5.1.6) the value was 2).

If we make the additional assumption that the partition consists of equally spaced intervals, then we can improve (5.1.6) to

$$(5.1.7) \quad ||e'|| \leq .7431 h ||f''|| .$$

Theorem 5.3 Let  $f$ ,  $f'$ ,  $f''$ , and  $f'''$  be continuous functions of period  $b - a$ . Then

$$(5.1.9) \quad ||e_i|| \leq (1/24) h^3 ||f'''|| ,$$

$$(5.1.10) \quad ||e_i'|| \leq (1/6) h^2 ||f'''|| ,$$

$$(5.1.11) \quad ||e|| \leq (1/24) h^3 ||f'''|| ,$$

$$(5.1.12) \quad ||e'|| \leq (7/24) h^2 ||f'''|| ,$$

$$(5.1.13)$$

$$||e''|| \leq [h_i/2 + (h^3/3h_i^2)] ||f'''|| , \quad x_{i-1} < x < x_i .$$

Marsden's analogous constants for (5.1.9) to (5.1.11) are  $1/8$ ,  $1/3$ ,  $17/96$ , and  $11/24$  respectively.

Furthermore, (5.1.9) and (5.1.11) are best possible. In fact we also have the exact pointwise bound

$$(5.1.14) \quad |e(x)| \leq |E_3(t)| h^3 ||f'''|| , \quad x_i \leq x \leq x_{i+1} ,$$

where  $t = (x - x_i)/(x_{i+1} - x_i)$  and

$$Q_3(t) = 1/24 - t^2/4 + t^3/6$$

is the "Euler spline" of degree 3.

The technique used here is the same as that used in the last chapter. For a given partition subinterval

$[x_i, x_{i+1}]$ , we write

$$(5.1.15) \quad |f^{(i)}(x) - s^{(i)}(x)| \leq |f^{(i)}(x) - L^{(i)}(x)| + |L^{(i)}(x) - s^{(i)}(x)|$$

where  $L$  is a polynomial interpolation of  $f$ . We then proceed by obtaining pointwise estimates of the quantities on the right hand side of (5.1.15).

### Proof of Theorem 5.1

Given that  $f$  and  $f'$  are continuously differentiable of period  $b - a$ , we will establish the following pointwise bound for the parabolic continuously differentiable spline  $s$  interpolating function values at subinterval midpoints  $z_i$ . Let  $x_i \leq x \leq x_{i+1}$  and let

$$t = (x - x_i) / (x_{i+1} - x_i) \text{ and } h = \max\{h_i\}.$$

Then for  $x_i \leq x \leq z_{i+1}$ , we have

$$(5.2.1) \quad |f(x) - s(x)| \leq h [1 + t - 8t^2 + 4t^3] ||f'||.$$

For  $z_{i+1} \leq x \leq x_{i+1}$  replace  $t$  in (5.2.1) by  $1 - t$ . Equation (5.1.3) follows from (5.2.1).

In order to establish (5.2.1) we write for

$$x_i \leq x \leq x_{i+1},$$

$$f(x) - s(x) = f(x) - L(x) + L(x) - s(x)$$

where  $L(x)$  is the parabola matching  $f(x_i)$ ,  $f(z_{i+1})$ , and  $f(x_{i+1})$ . Then

$$(5.2.2) \quad |f(x) - s(x)| \leq |f(x) - L(x)| + |L(x) - s(x)|.$$

We can represent  $L(x)$  as

$$(5.2.3) \quad L(x) = f(x_i)A_0(t) + f(z_{i+1})A_1(t) + f(x_{i+1})A_2(t)$$

where  $t = (x - x_i)/h_i$  and

$$A_0(t) = 2(1/2 - t)(1 - t),$$

$$A_1(t) = 4t(1 - t),$$

$$A_2(t) = 2t(t - 1/2).$$

As  $L$  reproduces parabolas exactly and as the restriction of  $s(x)$  to  $[x_i, x_{i+1}]$  is a parabola, for  $x_i \leq x \leq x_{i+1}$  we have

$$(5.2.4) \quad s(x) = s(x_i)A_0(t) + s(z_{i+1})A_1(t) + s(x_{i+1})A_2(t).$$

As  $f(z_{i+1}) = s(z_{i+1})$ , we have

$$(5.2.5) \quad \begin{aligned} |L(x) - s(x)| &\leq |f(x_i) - s(x_i)| |A_0(t)| \\ &\quad + |f(x_{i+1}) - s(x_{i+1})| |A_2(t)| \\ &\leq \{ |A_0(t)| + |A_2(t)| \} ||e_i|| \\ &\leq |1 - 2t| ||e_i|| \end{aligned}$$

where  $||e_i|| = \max_{1 \leq i \leq k} |f(x_i) - s(x_i)|$ .

We have shown that

$$(5.2.6) \quad |f(x) - s(x)| \leq |f(x) - L(x)| + |1 - 2t| ||e_i||.$$

It remains to bound  $|f(x) - L(x)|$  and  $||e_i||$  in terms of  $||f'||$ .

Marsden showed that

$$(5.2.7) \quad ||e_i|| \leq h ||f'||$$

where  $h$  is the maximum length of a subinterval.

In order to bound  $|f(x) - L(x)|$  we resort to the Peano theorem. Defining  $g(t) =: f(x_i + h_i t) = f(x)$ , we have

$$(5.2.8) \quad f(x) - L(x) = \int_0^1 K_1(t, z) g'(z) dz$$

where

$$\begin{aligned} K_1(t, z) = & (t-z)_+^0 - A_0(t) [0-z]_+^0 \\ & - A_1(t) [1/2 - z]_+^0 \\ & - A_2(t) [1 - z]_+^0 \end{aligned}$$

and

$$(t-z)_+^0 = \begin{cases} 1 & \text{for } t \geq z \\ 0 & \text{for } t < z. \end{cases}$$

In order to verify (5.2.8), one need only expand the right hand side and integrate by parts. For  $0 \leq t \leq 1/2$ ,  $K_1(t, z)$  may be written in the more convenient form

$$\begin{aligned} K_1(t, z) = & -A_1(t) - A_2(t) + 1 && \text{for } 0 \leq z \leq t \\ = & -A_1(t) - A_2(t) && \text{for } t \leq z \leq 1/2 \\ = & -A_2(t) && \text{for } 1/2 \leq z \leq 1. \end{aligned}$$

From Equation (5.2.8) it follows that

$$\begin{aligned} (5.2.9) \quad |f(x) - L(x)| & \leq \int_0^1 |K_1(t, z)| dz \max_{0 \leq t \leq 1} |g'(t)| \\ & \leq h_1 \int_0^1 |K_1(t, z)| dz \max_{x_i \leq x \leq x_{i+1}} |f'(x)| \\ & \leq h_1 \int_0^1 |K_1(t, z)| dz \|f'\|. \end{aligned}$$

Evaluating the integral in (5.2.9), we have

$$\begin{aligned} (5.2.10) \quad \int_0^1 |K_1(t, z)| dz = & t [1 - A_1(t) - A_2(t)] \\ & + (1/2 - t) [A_1(t) + A_2(t)] \\ & + (1 - 1/2) [-A_2(t)] \\ = & 3t - 8t^2 + 4t^3. \end{aligned}$$

Combining Equations (5.2.7)-(5.2.10) we have for

$$0 \leq t \leq 1/2,$$

$$(5.2.11) \quad |f(x) - s(x)| \leq \{h(1 - 2t) + h_i(3t - 8t^2 + 4t^3)\} ||f'|| \\ \leq h[1 + t - 8t^2 + 4t^3] ||f'||$$

which is precisely the desired result. The maximum of the right hand side of (5.2.11) occurs for  $a_0 = 2/3 - 13/6$ . Evaluating gives the value  $c_{0,1}$ .

#### Proof of Theorem 5.2

Let  $f$  be twice continuously differentiable of period  $b - a$  and let a partition

$a = x_0 < z_1 < x_1 < \dots < x_i < z_{i+1} < x_{i+1} < \dots < x_n = b$  be given (where  $z_{i+1} = (x_i + x_{i+1})/2$ , every  $i$ ). Let  $s$  be continuously differentiable and a parabola on each interval  $[x_i, x_{i+1}]$  such that

$$s(z_{i+1}) = f(z_{i+1}), s(a) = s(b), \text{ and } s'(a) = s'(b).$$

Letting  $t = (x - x_i)/h_i$ , we show that

$$(5.3.1) \quad |f(x) - s(x)| \leq c_{0,2}(t) ||f''|| \\ c_{0,2}(t) = h^2 \{(1 - 2t)/6 + [t/(3 - 2t) - t^2]\}$$

and for  $z_{i+1} \leq x \leq x_{i+1}$

$$c_{0,2}(t) = c_{0,2}(1-t).$$

Furthermore the maximum of  $c_{0,2}(t)$  is  $1/6$  and occurs for  $t = 0$  and  $1$ .

As in the proof of Theorem 5.1 we fix  $i$  and let  $L(x)$  be the parabola satisfying

$$L(x_i) = f(x_i) \quad , \quad L(z_{i+1}) = f(z_{i+1}) \quad ,$$

$$L(x_{i+1}) = f(x_{i+1}) \quad .$$

Then, proceeding in the same way as before,

$$(5.3.2) \quad |f(x) - s(x)| \leq |f(x) - L(x)| + ||e_i|| |1 - 2t| \quad .$$

We must bound  $|f(x) - L(x)|$  and  $||e_i||$ . We first bound  $||e_i||$ . From Marsden [1974], we have

$$(5.3.3) \quad h_i s_{i-1} + 3(h_i + h_{i-1}) s_i + h_{i-1} s_{i+1} \\ = 4 [h_i f(z_i) + h_{i-1} f(z_{i+1})] \quad .$$

Denoting  $f_i = f(x_i)$  and  $e_i = f_i - s_i$ , we obtain from Equation (5.3.3)

$$(5.3.4) \quad h_i e_{i-1} + 3(h_i + h_{i-1}) e_i + h_{i-1} e_{i+1} \\ = h_i f_{i-1} - 4 h_i f(z_i) + 3(h_i + h_{i-1}) f_i \\ - 4 h_{i-1} f(z_{i+1}) + h_{i-1} f_{i+1} \\ =: B(f) \quad .$$

As  $B$  is identically zero for any linear function  $f$ , we have by the Peano Theorem:

$$(5.3.5) \quad B(f) = \int_{x_{i-1}}^{x_{i+1}} K(y) f''(y) dy / 1!$$

where

$$K(y) = B_x[(x-y)_+] \\ = h_{i-1} (x_{i+1} - y)_+ - 4h_{i-1} (z_{i+1} - y)_+ \\ + 3(h_i + h_{i-1}) (x_i - y)_+ - 4h_i (z_i - y)_+ \\ + h_i (x_{i-1} - y)_+$$

and

$$(x - y)_+ = \begin{array}{ll} x - y & \text{for } x \geq y \\ 0 & \text{for } x < y \end{array} \quad .$$

In order to illustrate the symmetry of the kernel  $K(y)$  about  $x_i$ , we expand in terms of  $y - x_i$  to obtain

$$\begin{aligned}
 K(y) &= h_{i-1} [h_i - (y-x_i)] \\
 &\quad \text{for } h_i/2 \leq y-x_i \leq h_i \\
 &= h_{i-1} [3(y-x_i) - h_i] \\
 &\quad \text{for } 0 \leq y-x_i \leq h_i/2 \\
 &= h_i [-3(y-x_i) - h_{i-1}] \\
 &\quad \text{for } -h_{i-1}/2 \leq y-x_i \leq 0 \\
 &= h_i [h_{i-1} + (y-x_i)] \\
 &\quad \text{for } -h_{i-1} \leq y - x_i \leq -h_{i-1}/2
 \end{aligned}$$

where  $h_i = x_{i+1} - x_i$  and  $h_{i-1} = x_i - x_{i-1}$ . As is easily seen, the sign of  $K(y)$  changes at  $y = x_i + h_i/3$  and  $x_i - h_{i-1}/3$ .

From (5.3.5), it follows that

$$\begin{aligned}
 (5.3.6) \quad & |h_i e_{i-1} + 3(h_i + h_{i-1}) e_i + h_{i-1} e_{i+1}| \\
 & \leq \int_{x_{i-1}}^{x_{i+1}} |K(y)| dy \quad ||f''|| \\
 & \leq (h_i + h_{i-1}) h_i h_{i-1} ||f''||/3 .
 \end{aligned}$$

Let  $i$  be such that  $|e_i| = ||e_i||$ . Then

$$(5.3.7) \quad ||e_i|| \leq (1/6) h^2 ||f''|| ,$$

which is the desired bound on  $||e_i||$ .

We next bound  $|f(x) - L(x)|$  where  $L$  is the parabola matching  $f$  at  $x_i$ ,  $z_{i+1}$ , and  $x_{i+1}$ .  $L$  can be uniquely expressed as

$$\begin{aligned}
 (5.3.8) \quad L(x) &= f(x_i) A_0(t) + f(z_{i+1}) A_1(t) \\
 &\quad + f(x_{i+1}) A_2(t)
 \end{aligned}$$

where

$$A_0(t) = 2 (1/2 - t) (1 - t) ,$$

$$A_1(t) = 4t (1 - t) ,$$

$$A_2(t) = 2t (t - 1/2) .$$

Then, defining  $g(t) =: f(x_i + h_i t) = f(x)$  we have

$$(5.3.9) \quad f(x) - L(x) = \int_0^1 K_2(t, z) g''(z) dz , \quad t = (x - x_i)/h_i$$

where

$$\begin{aligned} K_2(t, z) = & (t - z)_+ - A_0(t) [0 - z]_+ \\ & - A_1(t) [1/2 - z]_+ - A_2(t) [1 - z]_+ . \end{aligned}$$

Equation (5.3.9) can be verified by integrating by parts to obtain (5.2.8). For  $0 \leq t \leq 1/2$ ,  $K_2$  takes the form

$$(5.3.10) \quad \begin{aligned} K_2(t, z) = & z (2t - 1) (1 - t) \text{ for } t \geq z, \quad t \leq 1/2, \\ & - t [1 + z(2t - 3)] \quad \text{for } t < z \leq 1/2 \\ & - t (2t - 1) (1 - z) \text{ for } z \geq 1/2, \quad t \leq 1/2 . \end{aligned}$$

From (5.3.9), it follows that for  $0 \leq t \leq 1/2$

$$\begin{aligned} (5.3.11) \quad |f(x) - L(x)| / |g''(t)| & \leq \int_0^1 |K_2(t, z)| dz \\ & = - \int_0^x z(2t - 1)(1 - t) dz \\ & \quad - \int_x^{1/(3-2t)} -t[1 + z(2t - 3)] dz \\ & \quad + \int_{1/(3-2t)}^{1/2} -t[1 + z(2t - 3)] dz \\ & \quad - \int_{1/2}^1 2t(t - 1/2)(1 - z) dz \\ & = -t^2 + [t/(3-2t)] . \end{aligned}$$

Therefore, if  $0 \leq t \leq 1/2$ , we have

$$(5.3.12) \quad |f(x) - L(x)| \leq h_i^2 \{-t^2 + [t/(3-2t)]\} ||f''(x)||$$

We can now assemble the parts to get the pointwise bound

(5.3.1). Using the bound for  $|f(x) - L(x)|$  of (5.3.11)

and the bound of (5.3.7) for  $||e_i||$  in the formula

$$|f(x) - s(x)| \leq |f(x) - L(x)| + ||e_i|| |1 - 2t|,$$

we then have for  $0 \leq t \leq 1/2$ ,

$$(5.3.13) \quad |f(x) - s(x)| \leq \{ h_i^2 [-t^2 + t/(3-2t)] + (1 - 2t) h^2/6 \} ||f''||$$

which, as  $h_i \leq h$ , immediately implies (5.3.1). The result for  $1/2 \leq t \leq 1$  follows by symmetry. It remains only to be shown that the maximum of

$$0 \leq c_{0,2}(t) = h^2 \{ (1 - 2t)/6 + [t/(3 - 2t) - t^2] \}$$

is  $h^2/6$  and occurs for  $t = 0$ . To see this, expand  $c_{0,2}(t)$  at 0 as

$$c_{0,2}(t) = c_{0,2}(0) + t c_{0,2}'(0) + (t^2/2) c_{0,2}''(y)$$

where  $0 \leq t \leq 1/2$  and  $0 \leq y \leq t$ . It is not hard to verify that the last two terms of the above expression are negative, and hence the maximum occurs at  $t = 0$ . This completes the proof of Equation (5.1.4).

We next show Equation 5.1.5. From Marsden, we have the tridiagonal system matching spline derivatives,

$$(5.3.14) \quad h_{i-1} s_{i-1}' + 3(h_i + h_{i-1}) s_i' + h_i s_{i+1}' \\ = 8 [f(z_{i+1}) - f(z_i)]$$

or equivalently,

$$\begin{aligned}
(5.3.15) \quad & h_{i-1} e_{i-1}' + 3(h_i + h_{i-1}) e_i' + h_i e_{i+1}' \\
& = h_{i-1} f_{i-1}' + 8f(z_i) + 3(h_i + h_{i-1}) f_i' \\
& \quad - 8f(z_{i+1}) + h_i f_{i+1}' \\
& =: B_1(f) . \\
& = \int_{x_{i-1}}^{x_{i+1}} B_1[(x-y)_+] f''(y) dy
\end{aligned}$$

where

$$\begin{aligned}
B_1[(x-y)_+] &= h_i & z_{i+1} \leq y \leq x_{i+1} \\
&= 8(y-x_i) - 3h_i & x_i \leq y \leq z_{i+1} \\
&= 8(y-x_i) + 3h_{i-1} & z_i \leq y \leq x_i \\
&= -h_{i-1} & x_{i-1} < y \leq z_i
\end{aligned}$$

and the last equality of (5.3.15) can be verified by integrating by parts and using the fact that  $B_1$  is identically zero if  $f$  is a linear function. From (5.3.15) follows

$$\begin{aligned}
(5.3.16) \quad & |h_{i-1} e_{i-1}' + 3(h_i + h_{i-1}) e_i' + h_i e_{i+1}'| \\
& \leq \int_{x_{i-1}}^{x_{i+1}} |B_1[(x-y)_+]| dy ||f''|| \\
& \leq (9/8) (h_i^2 + h_{i-1}^2) ||f''|| ,
\end{aligned}$$

and hence

$$(5.3.17) \quad ||e_i'| \leq (9/16) h ||f''|| ,$$

which is precisely (5.1.5).

We next establish a pointwise bound on  $|e'(t)|$ . We show that for  $x_i \leq x \leq x_{i+1}$ , and  $t = (x - x_i)/h_i$ ,

$$\begin{aligned}
(5.3.18) \quad |f'(x) - s'(x)| &\leq \{(9/16)h \\
&\quad + 2 h_i t(1-t)\} ||f''|| .
\end{aligned}$$

If we maximize the right hand side of (5.3.18) by taking  $h_i = h$  and  $t = 1/2$ , then (5.1.6) is an immediate consequence of (5.3.18).

To establish (5.3.18), we let  $J(x)$  be the unique parabola satisfying

$$(5.3.19) \quad J(z_{i+1}) = f(z_{i+1}), \quad J'(x_i) = f'(x_i), \\ J'(x_{i+1}) = f'(x_{i+1}).$$

Then

$$J'(x) - s'(x) = (1-t) e_i' + t e_{i+1}'$$

and

$$(5.3.20) \quad |f'(x) - s'(x)| \\ \leq |f'(x) - J'(x)| + |J'(x) - s'(x)| \\ \leq |f'(x) - J'(x)| + ||e_i'|| \{ |1-t| + |t| \} \\ \leq |f'(x) - J'(x)| + ||e_i'||.$$

As we already have an estimate for  $||e_i'||$ , we need only estimate  $|f'(x) - J'(x)|$ . It is easy to see by integration by parts that

$$(5.3.21) \quad f'(x) - J'(x) = \int_0^1 K(t, z) g''(z) dz / h_i$$

where  $g(t) =: f(x_i + h_i t) = f(x)$  and where

$$K(t, z) = \begin{array}{ll} 1 - t & \text{for } t \geq z \\ -t & \text{for } t < z, \end{array}$$

From (5.3.21) follows

$$(5.3.22) \quad |f'(x) - s'(x)| \leq \int_0^1 |K(z, t)| dz ||g''|| / h_i$$

and evaluation of (5.3.22) gives

$$(5.3.23) \quad |f'(x) - J'(x)| \leq h_i 2t (1 - t) ||f''||.$$

Applying (5.3.23) and (5.3.17) in (5.3.20) gives

$$(5.3.24) \quad |f'(x) - s'(x)| \leq \{h_i 2t (1 - t) + (9/16) h\} ||f''||,$$

which is the desired pointwise bound.

In order to improve the bound for the case of evenly spaced knots we return to the use of the parabola  $L$  satisfying

$$L(x_i) = f(x_i), \quad L(z_{i+1}) = f(z_{i+1}),$$

$$L(x_{i+1}) = f(x_{i+1}).$$

Then for arbitrary  $i$  and  $x_i \leq x \leq x_{i+1}$ , we have

$$(5.3.25) \quad |f'(x) - s'(x)| \leq |f'(x) - L'(x)| + |L'(x) - s'(x)| \\ \leq |f'(x) - L'(x)| + (1/h_i) |e_i A_0'(t) + e_{i+1} A_2'(t)| \\ \leq |f'(x) - L'(x)| + (||e_i||/h_i) \{|A_0'(t)| + |A_2'(t)|\}$$

where  $t = (x - x_i)/h_i$ . Differentiating equations (5.3.8) gives

$$(5.3.26) \quad |A_0'(t)| = |3 - 4t|,$$

$$|A_2'(t)| = |1 - 4t|.$$

Recalling from Equation (5.3.7) that

$$||e_i|| \leq (1/6) h^2 ||f''||,$$

we have

$$(5.3.27) \quad |f'(x) - s'(x)| \leq |f'(x) - L'(x)| + (h^2/6h_i) \{|A_0'(t)| + |A_2'(t)|\} ||f''||$$

where

$$\begin{aligned}
|A_0'(t)| + |A_2'(t)| &= 4 - 8t, \quad 0 \leq t \leq 1/4 \\
&= 2, \quad 1/4 \leq t \leq 3/4 \\
&= 8t - 4, \quad 3/4 \leq t \leq 1.
\end{aligned}$$

To obtain a pointwise bound we need only bound  $|f'(x) - L'(x)|$ . Differentiating (5.3.9), we obtain

(5.3.28)  $|f'(x) - L'(x)| \leq$

$$h_i \int_0^1 |K_2^{(1,0)}(t, z)| dz \quad ||f''(x)||$$

where for  $0 \leq t \leq 1/2$

$$\begin{aligned}
K_2^{(1,0)}(t, z) &= z(3 - 4t), \quad t \geq z \\
&= -1 + z(3 - 4t), \quad t < z \leq 1/2 \\
&= (1 - 4t)(1 - z), \quad 1/2 \leq z \leq 1.
\end{aligned}$$

The only sign changes occur for  $z = 1/(3 - 4t)$  and  $0 \leq t \leq 1/4$  and along the lines  $t = 1/4$ ,  $z \leq 1/2$ , and  $z = t$ . Evaluating the the integral of (5.3.28) for  $0 \leq t \leq 1/4$  then gives

$$\begin{aligned}
(5.3.29) \quad \int_0^1 |K_2^{(1,0)}(t, z)| dt &= \int_0^t z(3 - 4t) dz \\
&= \int_t^{1/(3-4t)} [-1 + z(3-4t)] dz \\
&+ \int_{1/(3-4t)}^{1/2} [-1 + z(3-4t)] dz \\
&+ \int_{1/2}^1 (1 - 4t)(1 - z) dz \\
&= t^2(3 - 4t) - 2t + 1/(3 - 4t).
\end{aligned}$$

For the interval  $1/4 \leq t \leq 1/2$ , the only sign change is the line  $z = t$ . We obtain

$$\begin{aligned}
 (5.3.30) \quad \int_0^1 |K_2^{(1,0)}(t,z)| dt &= \int_0^x z(3-4t) dz \\
 &- \int_x^{1/2} [-1 + z(3-4t)] dz \\
 &- \int_{1/2}^1 (1-4t)(1-z) dz \\
 &= t^2(3-4t) .
 \end{aligned}$$

The expression of (5.3.30) is monotone increasing for  $1/4 \leq t \leq 1/2$ . We have shown that

$$\begin{aligned}
 (5.3.31) \quad |f'(x) - L'(x)| / |f''| & \\
 & h_i \{t^2(3-4t) - 2t + 1/(3-4t)\} , \\
 & 0 \leq t \leq 1/4 \\
 \leq & h_i t^2(3-4t) , \\
 & 1/4 \leq t \leq 1/2 .
 \end{aligned}$$

Combining (5.3.27) and (5.3.31) we have

$$\begin{aligned}
 (5.3.32) \quad |f'(x) - s'(x)| / |f''| &\leq \\
 & h_i [t^2(3-4t) - 2t + 1/(3-4t)] \\
 & + (h^2/3h_i)(2-4t) , \\
 & 0 \leq t \leq 1/4 \\
 \leq & h_i [t^2(3-4t)] + h^2/3h_i , \\
 & 1/4 \leq t \leq 1/2 .
 \end{aligned}$$

As usual these results can be extended by symmetry to the interval  $1/2 \leq t \leq 1$ . The bound given in (5.3.32) is monotone decreasing from 0 to 1/4 and increasing from 1/4 to 1/2. For an equally spaced partition, and  $t$  near 1/2, (5.3.32) is quite a bit smaller than (5.3.23). For instance for  $t = 1/2$ , we have  $1/3 + 1/4$  versus  $17/16$  from (5.3.23). If we rewrite (5.3.23) as

$$(5.3.33) \quad |f'(x) - s'(x)| \leq C_{2,1}(t) ||f''||$$

where

$$C_{2,1}(t) = h_i 2t(1-t) + (9/16) h$$

and write (5.3.32) as

$$(5.3.34) \quad |f'(x) - s'(x)| \leq c_{2,1}(t) ||f''||$$

where  $c_{2,1}(t)$  is the right hand side of (5.3.32), then we have also the pointwise bound

$$(5.3.34) \quad |f'(x) - s'(x)| \leq \min\{C_{2,1}(t), c_{2,1}(t)\} ||f''||.$$

For  $h = h_i$ , the maximum of the right hand side of (5.3.34) is approximately .7431  $h ||f''||$  and occurs for  $t$  approximately equal to .10038.

This completes the proof of Theorem 5.2.

### Proof of Theorem 5.3

We first show that if  $f$ ,  $f'$ ,  $f''$ , and  $f'''$  are continuous and of period  $b - a$ , then

$$(5.4.1) \quad ||f - s|| \leq (1/24) h^3 ||f'''||$$

where  $s$  is the parabolic and periodic (once differentiable) spline interpolating subinterval midpoints. Furthermore, "1/24" cannot be improved.

We proceed by demonstrating (5.1.14). For a given partition and subinterval  $[x_i, x_{i+1}]$ , we write

$$(5.4.2) \quad |f(x) - s(x)| \leq |f(x) - L(x)| + |L(x) - s(x)|$$

where  $L(x)$  is the unique parabola satisfying

$$(5.4.3) \quad L(x_i) = f(x_i), \quad L(z_{i+1}) = f(z_{i+1}), \\ L(x_{i+1}) = f(x_{i+1})$$

where  $z_{i+1} = (x_{i+1} + x_i)/2$ .  $L(x)$  may be uniquely expressed as

$$(5.4.4) \quad L(x) = f(x_i) A_0(t) + f(z_{i+1}) A_1(t) + f(x_{i+1}) A_2(t)$$

where  $t = (x - x_i)/h_i$ , and

$$A_0(t) = 2(1/2 - t)(1 - t),$$

$$A_1(t) = 4t(1 - t),$$

$$A_2(t) = 2t(1/2 - t).$$

Proceeding by using the Cauchy formula one obtains

$$(5.4.5) \quad |f(x) - L(x)| \leq (1/6) |t(1/2 - t)(1 - t)| ||f'''||.$$

In order to bound

$$(5.4.6) \quad |L(x) - s(x)| = | [f(x_i) - s(x_i)] A_0(t) + [f(x_{i+1}) - s(x_{i+1})] A_2(t) | \\ \leq ||e_i|| \{ |A_0(t)| + |A_2(t)| \} \\ = ||e_i|| |1 - 2t|,$$

we must bound  $||e_i|| = \max_{j=1,2,\dots,k} \{ |e_j| \}$  where

$e_j = f(x_j) - s(x_j)$ . To bound  $||e_i||$ , we resort in turn to the tridiagonal system of (5.3.4),

$$(5.4.7) \quad h_i e_{i-1} + 3(h_i + h_{i-1}) e_i + h_{i-1} e_{i+1} \\ = h_i f_{i-1} - 4h_i f(z_i) + 3(h_i + h_{i-1}) f_i \\ - 4h_{i-1} f(z_{i+1}) + h_{i-1} f_{i+1} \\ =: B_1(f).$$

$B_1(f)$ , so defined, is a linear functional identically zero for polynomials of degree two or less. We thus have

$$(5.4.8) \quad B_1(f) = \int_{x_{i-1}}^{x_{i+1}} K(y) f'''(y) dy / 2!$$

where

$$\begin{aligned}
K(y) &= B_1[(x-y)_+^2] \\
&= h_{i-1} [x_{i+1} - y]_+^2 - 4h_{i-1} [z_{i+1} - y]_+^2 \\
&\quad + 3(h_i + h_{i-1}) [x_i - y]_+^2 \\
&\quad - 4h_i [z_i - y]_+^2 + h_{i-1} [x_{i-1} - y]_+^2 \\
&= h_{i-1} [h_i - (y-x_i)]^2, \quad h_i/2 \leq y-x_i \leq h_i \\
&\quad h_{i-1} [h_i - (y-x_i)]^2 - 4h_{i-1} [h_i/2 - (y-x_i)]^2, \\
&\quad 0 \leq y-x_i \leq h_i/2 \\
&\quad -h_i [h_{i-1} + (y-x_i)^2] + 4h_i [h_{i-1}/2 + (y-x_i)]^2, \\
&\quad -h_{i-1}/2 \leq y-x_i \leq 0 \\
&\quad -h_i [h_{i-1} + (y-x_i)]^2, \quad -h_{i-1} \leq y-x_i \leq -h_{i-1}/2.
\end{aligned}$$

Then

$$\begin{aligned}
(5.4.9) \quad & |h_i e_{i-1} + (h_i + h_{i-1}) e_i + h_{i-1} e_{i+1}| \\
& \leq \int_{x_{i-1}}^{x_{i+1}} |K(y)| dy \quad ||f'''|| / 2! \\
& = (1/12) (h_{i-1} h_i^3 + h_i h_{i-1}^3) ||f'''||.
\end{aligned}$$

If we take  $i$  so that  $|e_i|$  is maximal, then we have

$$\begin{aligned}
(5.4.10) \quad & 2(h_i + h_{i-1}) ||e_i|| \leq \\
& = (1/12) (h_{i-1} h_i^3 + h_i h_{i-1}^3) ||f'''||
\end{aligned}$$

and hence

$$\begin{aligned}
(5.4.11) \quad & ||e_i|| \\
& \leq (1/24) (h_{i-1} h_i^3 + h_i h_{i-1}^3) / (h_i + h_{i-1}) ||f'''|| \\
& \leq (1/24) h^3 ||f'''||.
\end{aligned}$$

Combining (5.4.6) and (5.4.11), we have

$$\begin{aligned}
(5.4.12) \quad & |f(x) - s(x)| \leq |f(x) - L(x)| + ||e_i|| |1 - 2t| \\
& \leq \{ |t(1/2-t)(1-t)|/6 + (1/24) |1-2t| \} h^3 ||f'''|| \\
& = |Q_3(t)| h^3 ||f'''||
\end{aligned}$$

where  $Q_3(t)$  is the Euler spline of degree three. Equation

(5.4.12) is precisely (5.1.14). From it follow also (5.1.9) and (5.1.11).

To see that (5.1.14) cannot be improved, consider the "Euler spline"  $Q_n(x)$  constructed by integrating a constant  $n$  times so that the  $n$ th integration is odd for  $n$  odd and even for  $n$  even. On the unit interval, the first few Euler splines are

$$\begin{aligned}(5.4.13) \quad Q_0(x) &= 1, \\ Q_1(x) &= x - 1/2, \\ Q_2(x) &= x^2/2 - x/2, \\ Q_3(x) &= x^3/6 - x^2/4 + 1/24.\end{aligned}$$

These can be compared to the even Euler polynomials given in Chapter Two.

If we extend the Euler splines to the real line by setting

$$(5.4.14) \quad Q_n(x) = (-1)^j Q_n(t+j)$$

$0 \leq t \leq 1$  and  $j$  integer, then  $Q_n(x)$  is  $n - 1$  times continuously differentiable and piecewise  $n$  times continuously differentiable.  $Q_n$  is of period 2 and has  $n$ th derivative of plus or minus one.  $Q_n$  is thus a member of the class of functions with  $n$ th derivative piecewise differentiable. The third derivative of  $Q_3$  can be represented as the pointwise limit of the the third derivative of a sequence  $\{f_i\}$  of three times continuously differentiable functions which converge uniformly to  $Q_3$ . Furthermore each of the  $f_i$  have third derivative bounded in absolute value by one.

Restricting  $Q_3(x)$  to any interval  $[0, 2k]$ , consider the once continuously differentiable spline  $s$  parabolic in each interval  $[i, i+1]$ , and satisfying

$$(5.4.15) \quad s(z_{i+1}) = Q_3(z_{i+1}) = 0, \\ s(0) = s(2k), \quad s'(0) = s'(2k).$$

The spline  $s$  thus defined is identically zero. It is not hard to see that the maximum error occurs at the integer knots and is  $1/24$ . In fact we have shown that  $Q_3(x)$  is actually a pointwise exact bound.

We next demonstrate Equation (5.1.10),

$$(5.1.10) \quad ||e_i'|| \leq (1/6) h^2 ||f'''||,$$

where  $||e_i'|| = \max_{0 \leq i \leq k} |f'(x_i) - s'(x_i)|$ . The proof uses the same functional  $B_1$  that we used in proving (5.1.5). As in that case we have

$$(5.4.16) \quad h_{i-1} e_{i-1}' + 3(h_i + h_{i-1}) e_i' + h_i e_{i+1}' \\ = h_{i-1} f_{i-1}' + 8f(z_i) + 3(h_i + h_{i-1}) f_i' \\ - 8f(z_{i+1}) + h_i f_{i+1}' \\ =: B_1(f),$$

which is identically zero for all polynomials of degree two or less. Hence

$$(5.4.17) \quad |B_1(f)| \leq \int_{x_{i-1}}^{x_{i+1}} |B_1[(x-y)_+^2]| dy \quad ||f'''||/2!$$

where

$$B_1[(x-y)_+^2] = 2h_i(x_{i+1} - y)_+ - 8(z_{i+1} - y)_+^2 \\ + 6(h_i + h_{i-1})(y - x_{i-1})_+ + 8(y - z_i)_+^2 \\ + 2h_{i-1}(y - x_{i-1})_+$$

$$\begin{aligned}
&= 2h_i^2 - 2h_i(y-x_i) , \\
&\quad h_i/2 \leq y-x_i \leq h_i \\
&6h_i(y-x_i) - 8(y-x_i)^2 , \\
&\quad 0 \leq y-x_i \leq h_i/2 \\
&-6h_{i-1}(y-x_i) - 8(y-x_i)^2 , \\
&\quad -h_{i-1}/2 \leq y-x_i \leq 0 \\
&2h_{i-1}^2 + 2h_{i-1}(y-x_i) , \\
&\quad -h_{i-1} \leq y-x_i \leq -h_{i-1}/2 .
\end{aligned}$$

Conveniently, the above kernel is positive. Evaluation of the integral in (5.4.17) is thus straightforward, leading to

$$\begin{aligned}
(5.4.18) \quad &|h_{i-1} e_{i-1}' + 3(h_i + h_{i-1}) e_i' + h_i e_{i+1}'| \\
&\leq (1/3) (h_i^3 + h_{i-1}^3) ||f'''|| .
\end{aligned}$$

Assuming that  $i$  is such that  $|e_i'|$  attains its maximum we then have

$$\begin{aligned}
(5.1.10) \quad &||e_i'|| \leq \frac{(h_i^3 + h_{i-1}^3)}{6(h_i + h_{i-1})} ||f'''|| \\
&\leq (1/6) h^2 ||f'''|| .
\end{aligned}$$

In order to extend the bound (5.1.10) to the entire interval, choose any subinterval  $[x_i, x_{i+1}]$  of the given partition and consider the line  $J'$  interpolating  $f_i'$  and  $f_{i+1}'$ .  $J'$  may be represented as

$$(5.4.19) \quad J'(x) = (1-t) f_i' + t f_{i+1}' .$$

By the triangle inequality, we have

$$\begin{aligned}
(5.4.20) \quad &|f'(x) - s'(x)| \leq |f'(x) - J'(x)| \\
&\quad + |J'(x) - s'(x)| .
\end{aligned}$$

As  $f'$  is twice continuously differentiable, we have the well-known inequality

$$(5.4.21) \quad |f'(x) - J'(x)| \leq h_i^2 t(1-t) ||f'''|| / 2! .$$

As both  $J'$  and  $s'$  are lines on  $[x_i, x_{i+1}]$ , we have

$$\begin{aligned} (5.4.22) \quad |J'(x) - s'(x)| &\leq [f_i' - s_i'] (1-t) + \\ &\quad [f_{i+1}' - s_{i+1}'] t \\ &\leq ||e_i'|| \{ |1-t| + |t| \} \\ &\leq ||e_i'|| \\ &\leq (1/6) h^2 ||f'''|| . \end{aligned}$$

Adding (5.4.21) and (5.4.22) gives the desired formula

$$\begin{aligned} (5.4.23) \quad |f'(x) - s'(x)| &\leq [1/6 + t(1-t)/2] h^2 ||f'''|| \\ &\leq (7/24) h^2 ||f'''|| . \end{aligned}$$

Several further refinements of this argument are possible. This particular bound may be worthy of further study in the future.

We next wish to bound  $f''(x) - s''(x)$ . Choosing the arbitrary partition subinterval  $[x_i, x_{i+1}]$ , we consider the parabola  $L$  matching  $f$  at  $x_i, z_{i+1}$ , and  $x_{i+1}$ . By the triangle inequality we have

$$\begin{aligned} (5.4.24) \quad |f''(x) - s''(x)| &\leq |f''(x) - L''(x)| + |L''(x) - s''(x)| \\ &\leq |f''(x) - L''(x)| \\ &\quad + (1/h_i^2) |e_i A_0''(t) + e_{i+1} A_2''(t)| \\ &\leq |f''(x) - L''(x)| \\ &\quad + (1/h_i^2) ||e_i|| \{ |A_0''(t)| + |A_2''(t)| \} \\ &\leq |f''(x) - L''(x)| \\ &\quad + (1/h_i^2) (h^3/24) ||f'''|| [4 + 4] \\ &\leq |f''(x) - L''(x)| + (h^3/3h_i^2) ||f'''|| . \end{aligned}$$

The bound on  $f''(x) - L''(x)$  is obtained by the Peano

theorem technique. We have for  $0 \leq t \leq 1$  and

$f \in C'''[x_i, x_{i+1}]$ , that

$$(5.4.25) \quad |f''(x) - L''(x)| \leq h_i \int_0^1 |K^{(2,0)}(t, z)| dz \quad ||f'''||/2!$$

where

$$\begin{aligned} K(t, z) &= (t - z)_+^2 - A_0(t) [0 - z]_+^2 \\ &\quad - A_1(t) [1/2 - z]_+^2 - A_2(t) [1 - z]_+^2 \\ &= (t - z)_+^2 - (1/2 - z)_+^2 4t(1 - t) \\ &\quad - (1 - z)^2 2t(t - 1/2) \end{aligned}$$

and for  $0 \leq t \leq 1/2$ ,

$$\begin{aligned} K^{(2,0)}(t, z)/2! &= 2z^2 & t \geq z \\ &= -1 + 2z^2 & t < z, z \leq 1/2 \\ &= -2(1 - z)^2 & t < z, z \geq 1/2. \end{aligned}$$

The first of these two terms is positive and the last two are negative. Evaluation of the integral of (5.4.25) is straightforward, giving for  $0 \leq t \leq 1/2$ ,

$$(5.4.26) \quad |f''(x) - L''(x)| \leq h_i \{ (4t^3/3) - t + 1/2 \} ||f'''||.$$

Using (5.4.26) in (5.4.24) gives for  $0 \leq t \leq 1/2$ ,

$$\begin{aligned} (5.4.27) \quad |f''(x) - s''(x)| &\leq \{ h_i [(4t^3/3) - t + 1/2] \\ &\quad + h^3/3h_i^2 \} ||f'''|| \\ &\leq \{ h_i/2 + h^3/3h_i^2 \} ||f'''||, \end{aligned}$$

which holds also for  $1/2 \leq t \leq 1$ .

Using the linear interpolation of  $f_i'$  and  $f_{i+1}'$  and the usual triangular inequality, we may obtain the alternate estimate

$$(5.4.28). \quad |f''(x) - s''(x)| \leq \{h_i[1/2 + t(1 - t)] + h^2/3h_i\} ||f'''||$$

which when  $h$  is larger than  $h_i$  sometimes offers a lower estimate of the error. As the proof is very similar to those already given, we omit the details.

## CHAPTER SIX CONCLUDING REMARKS

In the above chapters, we have given bounds for the error of approximation of several polynomial and spline interpolations. Rather than restate the theorems proved in previous chapters, we will try to indicate what further work is possible and desirable.

The work done here has provided some of the motivation for the work of Bojanov and Varma (in preparation) extending Cauchy's formula for the error of polynomial interpolation to an expression for derivative error. They proved the following theorem.

Theorem 6.1 (Bojanov and Varma) Let  $f \in C^{(n+1)}[a,b]$  and let  $L[f,x]$  be the polynomial of degree  $n$  interpolating  $f$  at  $n+1$  points. Then for  $i < n+1$ , we have

$$(6.1.1) \quad |f^{(i)}(x) - L^{(i)}[f,x]| \leq \frac{|f^{(n+1)}(x)|}{(n+1)!} \prod_{i=0}^n |x - x_i|$$

where

$$L_{n+1}(x) = \prod_{i=0}^n (x - x_i) .$$

Furthermore, (6.1.1) continues to hold for the case of Hermite interpolation (when  $L_n(x)$  is appropriately redefined). In particular, let  $f \in C^{(2m)}[0,1]$  and let  $v_{2m-1}$  by the two-point Hermite interpolation satisfying

$$v_{2m-1}(0) = f(0) , \quad v_{2m-1}(1) = f(1) ,$$

$$v_{2m-1}^{(i)}(0) = f^{(i)}(0) , \quad v_{2m-1}^{(i)}(1) = f^{(i)}(1) ,$$

for  $i = 1, 2, \dots, m-1$ . Then

$$(6.1.2) \quad |f^{(i)}(x) - v_{2m-1}^{(i)}(x)|$$

$$\leq \left| \frac{d^i}{dx^i} x^m(1-x)^m \right| |f^{(2m)}| / (2m)! .$$

Equation (6.1.2) generalizes the results of Birkhoff and Priver [1967]. A further generalization of (6.1.1) could hold for well-posed two-point Birkhoff interpolation and thus give the norm bounds of Chapters Two and Three.

An alternate approach, suggested in conversation with Garrett Birkhoff, is to compute pointwise bounds for the derivatives of polynomial interpolation by the automated approximate evaluation of Peano kernels. A computer routine that accomplishes the necessary evaluations is given by Howell and Diaa (available on request). Similar routines could automate some other sorts of error bounds. Some of the possibilities are detailed in Howell and Diaa.

We next discuss further possibilities for the study of spline error bounds. Hopefully, knowledge of the errors of such error bounds should aid in comparing various spline operators. For instance, for the class of functions three times continuously differentiable, the periodic parabolic splines were shown in Theorem 5.3 to have bounds very close to the cubic splines of the type discussed by Varma and Katsifarakis (Theorem 1.17). On the other hand, many of the parabolic spline bounds can be

bettered by a  $C^{(2)}$  quartic spline which will be discussed in later work.

Many other spline error bounds may be amenable to the same techniques employed here. Among these are the derivative bounds for the  $C^{(2)}$  quartic spline of Chapter Four, bounds for the lacunary quintics discussed by Meir and Sharma [1973], and bounds for the local scheme proposed by Prasad and Varma [1979].

Even the extensively studied cubic splines require further study along these lines. The techniques used here might yield good results when the interpolated functions are once or twice continuously differentiable. Even the second derivative bound given by Hall and Meyer [1976] for  $C^{(2)}$  cubic splines interpolating  $C^{(4)}$  functions differentiable is not exact. Finally of course it would be of interest to develop a method (even if merely numeric) of deriving optimal bounds for spline interpolation in each case that the problem makes sense. Another type of error problem which is accessible by techniques similar to the ones given here is the error of the quadrature associated with any given piecewise polynomial interpolation.

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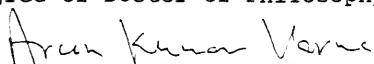
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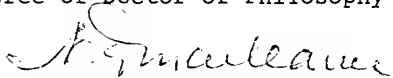
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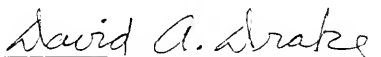
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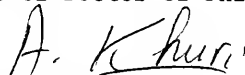
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